ON TOP CATEGORIES

BY

WAGISH SHUKLA

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

AUGUST 1971

ON TOP CATEGORIES

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

BY
WAGISH SHUKLA

to the
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR
AUGUST 1971

CERTIFICATE

It is to certify that the work embodied in the thesis "Top Categories" by Wagish Shukla has been carried out under my supervision and has not been submitted elsewhere for a degree.

S P Frank

(S.P. Franklin)

ACKNOWLEDGEMENTS

The author gratefully thanks:

1. Professor Oswald Wyler, who read an earlier draft of this thesis and corrected several errors, filled many gaps and suggested several improvements.

2. His teachers who have always encouraged him and taken a keen interest in his progress.

3. His friends, specially those in the SCKOWSTOMS group of I.I.T., Kanpur of which the author was a founder member.

4. I.I.T., Kanpur, for financial support while the author was a research scholar and Tata Institute of Fundamental Research, Bombay for awarding a visitng position during 1969-70.

5. Mr. D.R. Joshi for typing out the final manuscript.

The author must separately mention his indebtedness to his thesis adviser Professor S.P. Franklin. He has made several invaluable suggestions at every stage of the development of this thesis besides boosting up the frequently sagging morale and preventing and curing many unpardonable errors in language, style and content. In addition he has deeply influenced author's thinking on mathematics (and to some extent, life). It is impossible for the author to thank him in any conventional sense.

Wagish Shukla

(WAGISH SHUKLA)

C O N T E N T S

SYNOPSIS

For some time, the problem of presenting a single theory in which 'continuity' of functions is a worthwhile concept has attracted attention. Descriptive theories of this sort were given by Csaszar (syntopogeneous spaces) and Katetov (mero topic spaces). It soon became clear however, that the problem was more suited to categorical techniques and in late sixties Kennison, Husek and Wyler among others, gave categorical solutions.

This thesis is concerned with Wyler's solution viz. top categories. The introduction is a bird's eye view of the impact of category theory on mathematics in general and general topology in particular. The first chapter consists of a proof that Wyler's and Husek's solutions are equivalent. Included is a discussion of the relationship with the solutions of Kennison and Bentley. The second chapter is devoted to collecting general information about top categories. The third chapter is a study of the behaviour of functors between top categories which 'lift' functors between base categories. The fourth studies reflections and coreflections in top categories. The fifth chapter is not on top categories. It consists of two remarks, one proposing a definition of free objects in concrete categories and the other being a characteri-

zation of the category of sets. Finally the appendix is a survey of some recent information about TOP obtained by using categorical methods.

INTRODUCTION

When Steenrod christened category theory as "abstract nonsense", he clearly had no idea that some mathematicians would take it so literally as to dangle it before every demand that contemporary mathematics makes of them. Sad to say, however, the stale jokes about soft and hard mathematics have been once again revived and are repeated in the same dull monotony. No wonder, therefore, that MITCHELL [143] has been obliged to regard every kind of sophistication other than the mathematical one, as a disqualification for studying category theory. Of course, mostly the scoffing has come from persons whose acquintance with category theory is limited to knowing that Steenrod called it abstract nonsense, but as DIEUDONNE [34] has pointed out, it is to be considered whether such an attitude is compatible with any amount of intellectual honesty.

Happily, this is a minor and fast disappearing irritation. More and more icy stares have been obliged to blink before the increasing lustre of facts. The test of the pudding, as somebody put it, is in its eating and the science of mathematics has been gleefully gourmandising on category theory ever since it was introduced by EILENBERG and MacLANE [37]. From a mild appetizer, it has rapidly promoted itself to one of the most important musts on the menu. Indeed, the swiftness and thoroughness

with which it has capured mathematical fancy is something very rare in the history of mathematics. Let us take just a cursory look at the disciplines presently under the spell of category theory.

GROTHENDIECK [63] has changed algebraic geometry beyond recognition. That homological algebra is a subject to be studied by abelian categorists has been long recognized (cf. BUCHSBAUM's appendix to CARTAN and EILENBERG [24]) but BARR and BECK ([16],[17]) and DEDECKER [33] among others, have given it a wholly new perspective. Universal algebra has been reduced to a branch of category theory by DAVIS [32], LAWVERE [124], LINTON [129] and WALTERS [168]. Homotopy theory has been taken to general categories by FREYD [48], GABRIEL and ZISMAN [50], HELLER [71], PUPPE [155] and ZILBER [178]. Functors have been successfully used in non-standard analysis by BACSICH [5], in Galois theory by CHASE and SWEEDLER [28] and in distribution theory by POPA ([151],[152]). That solid connections have been established with functional analysis can be seen, among several others, in the survey by MITYAGIN and SHVARTS [144], in the expository articles by SEMADENI ([161],[162]) and the announcements by NEGREPONTIS [146] and WILBER [170]. HAJEK [64] and MANES [137] have found in it the proper language for topological dynamics, LAWVERE [126] for classical particle and fluid dynamics (with the same

hopes for quantum mechanics) and GRAVES [58] for combinatorial geometry. The categorical point of view has been adopted by LINTON [128] and JOYAL [102] to study measure theory and by HELLER [70] to study stochastic transformations. By using categories, FEFERMAN [40], KREIZEL [118], LAMBEK ([121],[122]), LAWVERE ([125],[127]) and MacLANE ([131],[133],[134]) have given a completely new outlook to mathematical logic and foundations. The same has been done for linguistics by BURGHELEA [23] and SCHNORR [159]. Automata theory has been firmly footed on categorical foundations by EILENBERG and WRIGHT ([38], [39]), GIVE'ON and ARBIB [52] and GOGUEN (unpublished). BAIANU [6] and ROSEN [157] have studied biology from the standpoint of category theory. Indeed, category theory is proving rather inadequate to cope with the growth of pure and applied sciences and "generalized nonsense" is now a necessary study. "Higher-dimensional nonsense" has been studied by BENABOU [19] and EILENBERG and KELLEY [36] not to mention the formidable work of EHRESMANN of which an exposition exists in [35]. One of the generalizations has been given by biologists; BAIANU [6] has defined 'super-categories'. 'Categoroids' introduced by KATETOV and FROLICK in [25] are another example. Some idea of what is in store for this futuristic discipline has been given by MacLANE [132].

The coverage in the paragraph above is far from exhaustive- the applications of category theory as a language and as more then just a language are sufficient to fill the hands of a full scale bibliographer. This is merely to point that the tail very much has a dog and that the wagging is by now mutual.

One of the areas where the applications have been of quite recent origin and the possibilities remain largely unexplored, is general topology. General topology, as used here, is a name for theories which are concerned with continuity of functions. Three main trends can be spotted.

Several attemptshave been made to define 'topological objects' in any category so that when one specializes to particular categories such as ENS, the category of sets and functions, one should get TOP, the category of topological spaces and continuous functions. There is EHRESMANN's theory of 'structuraed categories' [35] which assigns a 'topology' to any object in any category just as it gives 'groups', 'equivalence relations' etc. in any category. Then there is the theory of KIM and RATTRAY [116]. According to them, if $\mathcal{A}$ is any category and $A \in \mathcal{A}$ (for 'A is an object of $\mathcal{A}$'; the notation I believe is due to Lutzer), an $\mathcal{A}$-topology on A is a limit preserving functor $X: \mathcal{A}^{op} \to$ TOP (where $\mathcal{A}^{op}$ is the opposite category of $\mathcal{A}$) such that the following diagram of categories and

functors commutes.

$$\begin{tikzcd} & \mathcal{A}^{op} \arrow[dl, "X"'] \arrow[dr, "\mathcal{A}(-,A)"] & \\ \mathrm{TOP} \arrow[rr, "\text{forgetful}", "\text{functor}"'] & & \mathrm{ENS} \end{tikzcd}$$

With these objects, and natural transformations as morphisms, they have been able to obtain a category which satisfies our expectations. ISBELL [101] has defined a topological object in a category $\mathcal{A}$ as a limit preserving functor $TOP^{op} \to \mathcal{A}$. BECK [18] has examined constructions of algebraic topology in such 'topological theories'. These constructions which yield most of the categories in topological algebra, are part of what may be called 'topological semantics'.

There has been categorical investigation of TOP itself. Its categorical characterizations, reflective and coreflective subcategories etc. have been studied. The appendix surveys some of these results.

The problem of finding a category of sets with "continuity structures" which should include the categories of topological spaces, uniform spaces, and proximity spaces has been attacked several times. The syntopogeneous spaces of CSASZAR [31] and the merotopic spaces of KATETOV [110] are two examples of descriptive theories

which were advanced. A general approach to this problem has been given by KATETOV and FROLICK in the notes of [26]. Briefly, a J-structure on a set Z is a collection of Γ of nonempty subsets of Z such that $S \in \Gamma$, $S \subset S' \subset Z$ implies $S' \in \Gamma$ and $S \cup S' \in \Gamma$ implies $S \in \Gamma$ or $S' \in \Gamma$. A J-space is a pair (Z,Γ) where Γ is a J-structure on Z. A J-morphism between two J-spaces (Z,Γ) and (Z',Γ') is given by a function $f : Z \to Z'$ such that $S \in \Gamma$ implies $f\, S \in \Gamma'$. For any functor φ: ENS $\to$ ENS, a φ-space is a pair (X,Γ) where Γ is a J-structure on the set φX and is called a φ-structure on X. A φ-continuous function $f: (X,\Gamma) \to (X',\Gamma')$ is a function $f: X \to X'$ such that $\varphi f: (\varphi X,\Gamma) \to (\varphi X',\Gamma')$ is a J-morphism. If Q: ENS $\to$ ENS is the functor defined by $QX = X \times X$, closure spaces, proximity spaces and uniform spaces can be shown to be particular cases of Q-spaces. Syntopogeneous spaces and merotopic spaces are also φ-spaces for suitably chosen φ's. Another general approach has been provided by HEDRLIN, PULTR, and TRNKOVA [69]. For functors (covariant or contravariant) F_i: ENS $\to$ ENS and types Δ_i, $i = 1,2,,\ldots,n$, they define a category whose objects are systems $(X; r_1,\ldots, r_n)$ where each r_i is a relational system of the type Δ_i on the set $F_i X$. A morphism $f: (X; r_1,\ldots,r_n) \to (Y; s_1,\ldots, s_n)$ is a function $f: X \to Y$ such that $F_i f$ are $r_i s_i$ - completible for covariant F_i's and $s_i r_i$ - compatible

for contravariant ones. Choosing F_i' s and Δ_i' s in a suitable manner, categories of topological spaces, proximity spaces, uniform spaces, merotopic spaces, topological groups etc. can all be shown to be particular cases of such a category. This approach is also useful in algebra.

This thesis is primarily concerned with examining a concept which is due to WYLER and gives a solution to the problem above - namely, the concept of top categories. As will be shown in chapter one, top categories are the same as S-categories of HUSEK [84] and not very different from the pullback stripping functors of KENNISON [112]. Closely related are T-categories of BENTLEY [20] and K-categories of KRISHNAN [119]. Among all these, we believe, top categories are the best bet. This concept has increased, modified and improved our knowledge of general topology and topological algebra, and provided a new-perspective. Its definition is less cumborsome than other ones and for reasons that shall be given later, it is more palatable to a categorist. In addition, as can be seen by the list of examples in chapter two, its formulation is most suited to several categories that appear in many areas of modern applied mathematics such as automata theory and pattern recognition.

One must not, however, expect too much from top categories. It is a general maxim of mathematics that the more general a theory, the less deep its results. Top categories are no exception to this rule. To begin with, the category of Hausdorff spaces, where the theory of topological spaces is most rich, is not a top category in any reasonable sense, nor are T_0-spaces, or for that matter, any of the usual separated categories. Indeed, as is evident from the definition of top categories, about the only thing which makes TOP a top category over ENS is the fact that the set of all topologies on a set is a complete lattice. Concepts like 'open set', 'closed set' etc. are not readily available in top categories. In fact, the comfortable manner in which most of the basic data of the underlying category-epimorphisms, monomorphisms, limits, bicategories etc. - lifts into the top category is itself, in some measure, a pointer.

But let us not forget - these difficulties are caused not so much by the solution proposed but by the problem itself; a theory which claims to include practically all continuity structures currently used in general topology and several in systems theory cannot possibly aspire to carry over the full richness of the theories it generalizes. It <u>is</u> possible to have a number of

fruitful constructions in top categories. In particular, the study of lifting functors seems to be distinctly promising.

The general plan of the thesis is as follows. Chapter one, titled 'Four approaches to categorical topology consists of a proof that top categories and S-categories are equivalent and a discussion of the relationship with pullback stripping functors and T-categories. Chapter two, titled 'Top categories' gives examples of top categories and studies a number of properties which can be lifted from the underlying category to the top category. These include, among others, constant-generation and bicategory structures in the sense of ISBELL. Chapter three, titled 'Lifting Functors', studies the behaviour of functors between two top categories which 'lift' functors between the base categories. Chapter four, titled 'Reflections and Coreflections' studies reflective and coreflective subcategories of top categories on the basis of the data provided by the base category. Chapter five, titled 'Free and Projective Objects' is not in tune with the rest of the thesis although a trivial connection can be established. It is more in the nature of an appendix and gives a theory of free and projective objects and a characterization of ENS.

Finally, the appendix surveys some aspects of application of category theory to the study of TOP.

Chapters have been divided into sections. Lemmas, theorems, propositions etc. in each section have been numbered A, B, ... in the order of their appearance. The material of every chapter is preceded by a brief introductory paragraph giving a summary of what follows.

CHAPTER – I

FOUR APPROACHES TO CATEGORICAL TOPOLOGY

In this chapter we compare HUSEK's S- categories and WYLER's top categories and show that they are the same. We also discuss the relationship of top categories with the pullback stripping functors of KENNISON and the T-categories of BENTLEY. Section one gives the definitions of these four concepts. These are essentially the same as given by the authors but in the case of T-categories sup and inf have been interchanged to suit out convenience. Section two consists of the proof that top categories and S-categories are the same. Section three briefly compares top categories to pullbackstripping functors and T-categories. We close with a discussion of why we select top categories for our study in the following chapters.

1.1 Definitions

Definition A : Top Categories

Let $\mathcal{A}$ be a category. A topological theory on $\mathcal{A}$ is a functor $p: \mathcal{A}^{op} \to ORD$ (where ORD is the category of partly ordered sets and orderpreserving functions) such that, for every $A \in \mathcal{A}^{op}_{o}$, pA is a complete lattice and for any morphism $f: B \to A$ in $\mathcal{A}^{op}$ (i.e., for any morphism $f: A \to B$ in $\mathcal{A}$, $pf: pB \to pA$ preserves infima. We usually write f^p in place of pf.

For a given topological theory p on $\mathcal{A}$, we construct a category $\underline{\mathcal{A}}^p$ as follows.

1. Objects are pairs (A, x) where $A \in \mathcal{A}_o$ and $x \in pA$.

2. A morphism (x, f, y): $(A, x) \to (B, y)$ or, more simply, f: $(A, x) \to (B, y)$ is given by a morphism $f: A \to B$ in $\mathcal{A}$ such that $x \leqslant f^p y$. Composition is lifted from $\mathcal{A}$.

It is clear that the objects and morphisms above do form a category. The category $\mathcal{A}^p$ is called a <u>top category</u> over $\mathcal{A}$ and may be thought of as a model of the topological theory p on $\mathcal{A}$. We usually refer to $\mathcal{A}$ as the base category. The elements of pA are said to be <u>structures</u> on A. The least element of pA, denoted by α_A (or α_A^p, if the situation demands it) is called the <u>discrete structure</u> on A. Similarly, the greatest element of pA, denoted by ω_A, is called the <u>indiscrete structure</u> on A. There is an obvious forgetful functor $P: \mathcal{A}^p \to \mathcal{A}$ given by $P(A, x) = A$ and $P(x, f, y) = f$

Top categories are due to WYLER and the above definition is given mostly in the terminology of his expository article [173]. His survey [172] is a comprehensive introduction to the topic and rather than his original papers we shall make [172] as our standard reference on top categories. [172] also includes some results from

an earlier draft of this thesis.

Definition B: $\mathcal{S}$- categories

A category $\mathcal{C}$ is called an S-category over a category $\mathcal{A}$ with respect to a functor $\Gamma : \mathcal{C} \to \mathcal{A}$ if the following five conditions hold:

S1. Γ is faithful.

S2. For any morphism f of $\mathcal{A}$, $\Gamma^{-1}[f]$ is non-empty. Moreover, if $\Gamma X = A$, $\Gamma Y = B$, $f \in \mathcal{A}(A,B)$ then there are morphisms $\varphi \in \Gamma^{-1}[f]$, $\psi \in \Gamma^{-1}[f]$ such that dom $\varphi = X$, codom $\psi = Y$.

S3. If φ is a morphism in $\mathcal{C}$ and $\Gamma\varphi = fg$ then there are morphisms $\varphi_1 \in \Gamma^{-1}[f]$, $\varphi_2 \in \Gamma^{-1}[g]$ such that $\varphi = \varphi_1\varphi_2$.

S4. For each $A \in \mathcal{A}_o$, the class $\Gamma^{-1}[A]$ is a set which is a complete lattice with respect to the following order:

$$X \leqslant Y \text{ iff } \Gamma\varphi = 1_A \text{ for some } \varphi \in \mathcal{C}(X,Y)$$

Note that such a φ, if it exists, is obviously unique, since Γ is faithful. Therefore the definition of the order does not depend on the choice of φ.

S5. If $\{\varphi_i | i \in I\}$ is a non-empty family of morphisms of $\mathcal{C}$ such that $\Gamma\varphi_i = \Gamma\varphi_j$ for every $(i,j) \in I \times I$, then there are morphisms

$$\varphi \in \mathcal{C}\,(\sup \operatorname{dom} \varphi_i,\ \sup \operatorname{codom} \varphi_i),$$
$$\text{and} \quad \psi \in \mathcal{C}\,(\inf \operatorname{dom} \varphi_i,\ \inf \operatorname{codom} \varphi_i)$$

such that $\Gamma\varphi = \Gamma\psi = \Gamma\varphi_i$ for all $i \in I$. Clearly φ and ψ are unique.

S- categories are due to HUSEK [84]. The bibliography lists some of his papers in which the power of S-categories has been demonstrated.

Definition C Pullback stripping functors

Let $H: \mathcal{C} \to \mathcal{A}$ be a faithful functor. Then for any $A \in \mathcal{A}_o$, $H^{-1}[A]$ is a partly ordered class with the following order:

$$X_1 \leqslant X_2 \text{ iff there is a } f \in \mathcal{C}\,(X_1, X_2) \text{ with } Hf = 1_A.$$

If $HX = A$, $HX' = A'$, let us call a morphism $\alpha: A \to A'$ admissible from X to X' iff there exists $g: X \to X'$ with $Hg = \alpha$. Then the order in $H^{-1}[A]$ is given by: $X_1 \leqslant X_2$ iff 1_A is admissible from X_1 to X_2.

Let now $\mathcal{C}$ be a productive category. A faithful functor $H: \mathcal{C} \to \mathcal{A}$ is a pullback stripping functor (or a psf) iff:

1. For each $A \in_o \mathcal{A}$, $H^{-1}[A]$ is productive and has a small skeleton.
2. For each $\alpha: A \to A'$ in $\mathcal{A}$ and each $X' \in H^{-1}[A']$ there exists a largest object X in $H^{-1}[A]$ such that α is admissible from X to X'. X is the α-pullback of X' and is denoted by $X = \alpha^o X'$.
3. $\alpha'^o \; \alpha^o X' = (\alpha\alpha')^o X'$, when defined.

Pullback stripping functors were defined by KENNISON [112]. He has not used them for any thing except TOP and its subcategories but as will be seen below, they are nearly as good as top categories or S-categories.

<u>Definition D</u> T-<u>categories</u>

Let $(\underline{L}, \leqslant)$ be a (possibly large) lattice in which every non-empty set has a sup and every set has an inf and such that there exists a (necessarily unique) element sup $\underline{L}$. A function $\Gamma: \underline{L} \to \mathrm{Ob}\ \mathrm{ENS}$ is said to be compatible with $\leqslant$ iff the following conditions hold:

i) If $\underline{X}$ is a subset of $\underline{L}$ then

$$\Gamma(\inf \underline{X}) = \dot{\bigcup_{X \in \underline{X}}} \Gamma X$$

ii) If $\underline{X}$ is a non-empty subset of $\underline{L}$ and if Γ is constant on $\underline{X}$ then $\Gamma X = \Gamma(\sup \underline{X})$ for every $X \in \underline{X}$

iii) $A \in_o \mathrm{ENS}$ implies that $\{X \in \underline{L} | \Gamma X \subset A\}$ is a set.

iv) sup $\underline{L} \neq X$ implies that ΓX is non-empty.

A concrete category $(\mathcal{L}, \Gamma)$ where $\Gamma : \mathcal{L} \to$ ENS is the faithful functor, is said to be a T-<u>category</u> iff the following nine conditions are satisfied.

T1. There is an order $\leqslant$ defined on the objects of $\mathcal{L}$ such that $(\text{ob}\, \mathcal{L}, \leqslant)$ is a lattice of the kind just described.

T2. Γ is compatible with $\leqslant$.

T3. For any $X \varepsilon \underset{o}{\mathcal{L}}$ and any function α, there exists an object denoted by $\alpha^{-1}[X]$ of $\mathcal{L}$.

T4. $\Gamma\, \alpha^{-1}[X] = \alpha^{-1}[\Gamma X]$

T5. If A is any set, α is a function and $X \varepsilon \underset{o}{\mathcal{L}}$ with $\alpha^{-1}[\Gamma X] \subset A$, and if $b = \alpha | A$ then $\alpha^{-1}[X] = b^{-1}[X]$

T6. If $X \varepsilon \underset{o}{\mathcal{L}}$ and α is the identity function on ΓX then $\alpha^{-1}[X] = X$

T7. $f : X \to Y$ is in $\mathcal{L}$ iff $X \leqslant (\Gamma f)^{-1}[Y]$

T8. If $\underline{X}$ is a subset of $\mathcal{L}$ then

$$\alpha^{-1}[\inf \underline{X}] = \inf \{ f^{-1}[X] \mid X \varepsilon \underline{X} \}$$

T9. If A, A' are sets, $\alpha : A \to A'$ is a function and $X = \sup\{L \underset{o}{\varepsilon} \mathcal{L} \mid \Gamma L = A\}$, $X' = \sup\{L \underset{o}{\varepsilon} \mathcal{L} \mid \Gamma L = A'\}$ then $X = \alpha^{-1}[X']$

T-categories were defined by BENTLEY [20]. Unfortunately, it seems that apart from TOP, no other nice example is available.

1.2 Top categories are the same as S-categories

Let $\mathcal{C}$ an S-category over $\mathcal{A}$ with respect to the functor $\Gamma : \mathcal{C} \to \mathcal{A}$. Then by S4, we know that for any $A \in \mathcal{A}_o$, $\Gamma^{-1}[A]$ is a complete lattice. We set $pA = \Gamma^{-1}[A]$ for an object A of $\mathcal{A}$. If $\alpha : A \to A'$ is a morphism in $\mathcal{A}$, we define $\alpha^p : pA' \to pA$ by setting

$$\alpha^p X' = \sup\{X \in pA \mid \Gamma\varphi = \alpha \text{ for some } \varphi \in \mathcal{C}(X, X')\}$$

for $X' \in pA'$. Note that by S2, $\Gamma^{-1}(\alpha)$ is non-empty and there is at least one $X \in pA$ and one $\varphi \in \mathcal{C}(X, X')$ such that $\Gamma\varphi = \alpha$. Since Γ is faithful, there is at most one such φ once an X has been found. If now X_1' and X_2' are two elements in pA' and $X_1' \leqslant X_2'$ then this means that there exists some $\varphi' \in \mathcal{C}(X_1', X_2')$ such that $\Gamma\varphi' = 1_{A'}$. Let us now look at the elements $\alpha^p X_1'$ and $\alpha^p X_2'$ of pA. By definition,

$$\alpha^p X_1' = \sup\{X \in pA \mid \Gamma\varphi = \alpha \text{ for some } \varphi \in \mathcal{C}(X, X_1')\}$$

$$\alpha^p X_2' = \sup\{X \in pA \mid \Gamma\varphi = \alpha \text{ for some } \varphi \in \mathcal{C}(X, X_2')\}$$

However, any time there exists a morphism $\varphi : X \to X_1'$, there surely exists a morphism $\varphi'\varphi : X \to X_2'$ where $\varphi' : X_1' \to X_2'$ such that $\Gamma\varphi' = 1_A'$, and if $\Gamma\varphi = \alpha$ then $\Gamma(\varphi'\varphi) = (\Gamma\varphi')(\Gamma\varphi) = 1_{A'}\alpha = \alpha$. Therefore, $\alpha^p X_1' \leqslant \alpha^p X_2'$ and α^p is indeed a morphism in ORD. Next, let us consider

a composition $A \xrightarrow{\alpha} A' \xrightarrow{\alpha'} A''$. If $X'' \varepsilon\, pA''$, then by definition,

$$(\alpha'\alpha)^p X'' = \sup\{X \varepsilon\, pA | \Gamma\varphi = \alpha'\alpha \text{ for some } \varphi \varepsilon\, \mathfrak{F}(X, X'')\}$$

To see that this is actually $\alpha^p \alpha'^p X''$, let $X \varepsilon\, pA$ such that $\Gamma\varphi = \alpha'\alpha$ for some $\varphi \varepsilon\, \mathfrak{F}(X, X'')$. Then by S3, there exist $X' \varepsilon\, pA'$ and $\varphi_1 : X' \to X''$, $\varphi_2 : X \to X'$ such that $\Gamma\varphi_1 = \alpha'$, $\Gamma\varphi_2 = \alpha$ and $\varphi = \varphi_1\varphi_2$. By definition,

$$\alpha'^p X'' = \sup\{Y' \varepsilon\, pA | \Gamma\psi = \alpha' \text{ for some } \psi \varepsilon\, \mathfrak{F}(Y', X'')\}$$

and hence $X' \leqslant \alpha'^p X''$. Since α^p is order-preserving, $\alpha^p X' \leqslant \alpha^p \alpha'^p X''$. But again,

$$\alpha^p X' = \sup\{Y \varepsilon\, pA | \Gamma\theta = \alpha \text{ for some } \theta \varepsilon\, \mathfrak{F}(Y, X')\}$$

and hence $X \leqslant \alpha^p X'$. Therefore $X \leqslant \alpha^p \alpha'^p X''$. This implies that $(\alpha'\alpha)^p X'' \leqslant \alpha^p \alpha'^p X''$. However, by S5, there exists $\theta_1 : \alpha'^p X'' \to X''$, $\theta_2 : \alpha^p \alpha'^p X'' \to \alpha'^p X''$ such that $\Gamma\theta_1 = \alpha'$, $\Gamma\theta_2 = \alpha$ so that $\Gamma(\theta_1\theta_2) = \alpha'\alpha$ and $\alpha^p \alpha'^p X'' \leqslant (\alpha'\alpha)^p X''$. Thus $(\alpha'\alpha)^p = \alpha^p \alpha'^p$. Moreover, if $X \varepsilon\, pA$ then $(1_A)^p X = \sup\{Y \varepsilon\, pA | \Gamma\varphi = 1_A \text{ for some } \varphi \varepsilon\, \mathfrak{F}(Y, X)\}$ $= \sup\{Y \varepsilon\, pA |\, Y \leqslant X\} = X$. Thus we have obtained a functor $p : \mathcal{A}^{op} \to \mathrm{ORD}$. In particular, if $\alpha : A \to A'$ is in $\mathcal{A}$, the definition of $\alpha^p X'$ for $X' \varepsilon\, pA'$ and S5 together imply that there exists a (necessarily unique) $\psi : \alpha^p X' \to X'$ with $\Gamma\psi = \alpha$. Therefore using the definition

of order in pA, we conclude that $X \leqslant \alpha^p X'$ iff there exists a (necessarily unique) $\varphi\colon X \to X'$ with $\Gamma\varphi = \alpha$. To see that all maps α^p preserve infima, let $\alpha\colon A \to A'$ be in $\mathcal{A}$, $\{X_i' \mid i \in I\} \subset pA'$, $X' = \inf X_i'$ and $X = \inf \alpha^p X_i'$. Then there are morphisms $\varphi_i\colon \alpha^p X_i' \to X_i'$ such that $\Gamma\varphi_i = \alpha$. By S5, there is one $\varphi\colon X \to X'$ such that $\Gamma\varphi = \alpha$. Therefore, $X \leqslant \alpha^p X'$. Next, assume that $Y \in pA$ is such that there exists $\eta\colon Y \to X'$ with $\Gamma\eta = \alpha$. However, $X' \leqslant X_i'$ for every i. Therefore, there exist $\beta_i\colon Y \to X' \to X_i'$ with $\Gamma\beta_i = \alpha$. In otherwords, $Y \leqslant \alpha^p X_i'$ for every i and hence $Y \leqslant X$. This implies that $\alpha^p X' \leqslant X$. Therefore, $X = \alpha^p X'$ and α^p preserves infima.

We now have a topological theory $p\colon \mathcal{A}^{op} \to CRD$ and the construction of the top category $\mathcal{A}^p$ is obvious. Its objects are pairs (A, X) where $A \in \mathcal{A}_o$ and $X \in \Gamma^{-1}[A]$. A morphism $\alpha\colon (A, X) \to (A', X')$ in $\mathcal{A}^p$ is given by a morphism $\alpha\colon A \to A'$ in $\mathcal{A}$ such that $X \leqslant \alpha^p X'$ i.e. such that there exists a $\varphi\colon X \to X'$ in $\mathcal{G}$ with $\Gamma\varphi = \alpha$. We now define a functor $E\colon \mathcal{G} \to \mathcal{A}^p$ as follows. For $X \in \mathcal{G}_o$, $EX = (\Gamma X, X)$. If $\varphi\colon X \to X'$ is a morphism in $\mathcal{G}$ then $E\varphi = \Gamma\varphi\colon (\Gamma X, X) \to (\Gamma X', X')$. Clearly, E is well defined. Since Γ is faithful, so is E. Also, if $\alpha\colon (A, X) \to (A', X')$ is in $\mathcal{A}^p$, then by earlier observations, there exists $\varphi\colon X \to X'$ with $E\varphi = \alpha$. Thus E is full. Finally, if

$(A,X) \in \mathcal{A}_o^P$, clearly, $EX = (A,X)$. Hence E is representative. Being full, faithful and representative, E is an equivalence. In fact, E is more than that, it is an isomorphism. In other words, $\mathcal{A}$ and $\mathcal{A}^P$ are the same.

To prove the converse of this result (and for many other purposes as well) we need the well known fact: *If A and B are complete ordered sets and $f: A \to B$ is in ORD, then f preserves infima iff there is a morphism $g: B \to A$ in ORD such that $y \leqslant fx$ holds iff $gy \leqslant x$, for all $x \in A$, $y \in B$.* Assume now that we are given a top category $\mathcal{A}^P$ with the faithful projection functor $P: \mathcal{A}^P \to \mathcal{A}$. We show that $\mathcal{A}^P$ is an S-category over $\mathcal{A}$ with respect to P. S1, saying that P is faithful, is satisfied. If $\alpha: A \to A'$ is in $\mathcal{A}$ then for any $x' \in pA'$, $\alpha: (A,\alpha_A) \to (A',x')$ is in $\mathcal{A}^P$ and thus $P^{-1}(\alpha)$ is non-empty for any $\alpha \in \mathcal{A}$. Also, if $P(A,x) = A$, $P(A',x') = A'$ and $\alpha: A \to A'$ is in $\mathcal{A}$, then $\alpha: (A, \alpha^P x') \to (A',x')$ is in $P^{-1}(\alpha)$ such that its codomain is (A',x'), and $\alpha: (A, x) \to (A', gx)$ is in $P^{-1}(\alpha)$ such that its domain is (A, x) where the existence of $g: pA \to pA'$ is guaranteed, by the fact just quoted above, because α^P preserves infima. This means that S2 is satisfied. Next, if $\alpha: (A,x) \to (A',x')$ is in $\mathcal{A}^P$ and $A \xrightarrow{\alpha} A' = A \xrightarrow{\alpha_1} A'' \xrightarrow{\alpha_2} A'$, then $x \leqslant \alpha^P x' = (\alpha_2\alpha_1)^P x' = \alpha_1^P \alpha_2^P x'$ so that $\alpha_1: (A, x) \to (A'', \alpha_2^P x')$ is in $P^{-1}(\alpha_1)$, $\alpha_2 : (A'',\alpha_2^P x') \to (A',x')$ is

in $P^{-1}(\alpha_2)$ and $(A,x) \xrightarrow{\alpha} (A',x') = (A,x) \xrightarrow{\alpha_1} (A'',\alpha_2^P x') \xrightarrow{\alpha_2} (A',x')$. Thus S3 is satisfied. Again, if $A \in \mathcal{A}_o$ then $P^{-1}[A]$ is certainly a set. The order defined by $(A,x_1) \leqslant (A, x_2)$ iff $1_A: (A,x_1) \to (A, x_2)$ is in $\mathcal{A}^P$ i.e. iff $x_1 \leqslant x_2$, makes $P^{-1}[A]$ a complete lattice and S4 is satisfied. Finally, if $\{\alpha: (A, x_i) \to (A', x_i') \mid i \in I\}$ is a non-empty family of morphisms of $\mathcal{A}^P$ then $x_i \leqslant \alpha^P x_i'$ so that $\sup x_i \leqslant \sup \alpha^P x_i'$ and $\inf x_i \leqslant \inf \alpha^P x_i'$. That is, $\alpha: (A, \sup x_i) \to (A', \sup x_i')$ and $\alpha: (A, \inf x_i) \to (A', \inf x_i')$ are also in $\mathcal{A}^P$.

We summarize our discussion in the following

<u>Theorem</u> <u>Assume that $\mathcal{K}$ is an S - category over a category $\mathcal{A}$ with respect to a functor</u> $P: \mathcal{K} \to \mathcal{A}$. <u>Then K is (isomorphic to) a top category over $\mathcal{A}$. Conversely, if $\mathcal{A}^P$ is a top category then it is an S-category over $\mathcal{A}$ with respect to its projection functor</u> P.

One consequence of this result is of some theoretical interest from the point of view of top categories. A top category, as defined, is only artificially invariant: A top category is one which is isomorphic to a constructed category $\mathcal{A}^P$. An S-category, however, is invariant. The observation that a top category is actually an S-category removes this shortcoming of the definition of a top category. Another method, which is a little more satisfactory

since it embeds the theory of top categories in the theory of fibered categories has been given by WYLER [172]. In that language, a top category $\mathcal{A}^p$ is a fibration $\mathcal{A}^p \to \mathcal{A}$ which is also an opfibration and has complete lattices as fibers.

1.3 <u>Relation between top categories, pullback stripping functors and T-categories.</u>

In the first place, it is obvious that if $\mathcal{A}^p$ is a top category over $\mathcal{A}$ then the projection P: $\mathcal{A}^p \to \mathcal{A}$ is a pullback stripping functor. In the converse direction, the following construction can be carried out. Let H: $\mathcal{C} \to \mathcal{A}$ be a pullback stripping functor and let $A \in \mathcal{A}_o$. Then $H^{-1}[A]$ has a small skeleton which is a complete lattice. Let this be denoted by pA. If α: A $\to$ A' is in $\mathcal{A}$ let α^p: pA' $\to$ pA be given by $\alpha^p = \alpha^o$. This association does define a functor p: $\mathcal{A}^{op} \to$ CRD (cf. KENNISON [112]). However, the author has not been able to decide whether the fact that each α^p preserves infima is built into KENNISON's axioms, in other words, whether a pullback-stripping functor always gives rise to a topological theory. According to WYLER [172], these two concepts are 'almost' equivalent; if 'almost' be interpreted as asking for an additional axiom about the preservation of infima then the equivalence is obvious. In this connection, clearly

one wants to talk about a stronger version of pullback stripping functors by knocking off the assumption that $\mathcal{A}$ is productive and then specifically requiring that each $H^{-1}[A]$ be complete.

For T-categories, one is almost tempted to say that top categories over ENS and T-categories are the same. Earlier, the author believed that under a mild assumption, this was in fact so, but Prof. Wyler, in a private communication has disproved it. The conjecture fails mainly on two counts: first, contravariance of the counter image operation does not seem to be built in the axioms for T-categories, and secondly, at least with the author's construction of $\mathcal{A}^p$ as a huge lattice, axiom T8 does not work. This construction was motivated by BENTLEY's [20] presentation of uniform spaces as T-categories which breaks down at precisely this point. In fact, as Prof. Wyler points out, it is beginning to be doubtful whether any reasonable example of a T-category, other than TOP, exists.

The arguments in this chapter make it clear that these four approaches are quite closely related and three of them are either equivalent or nearly so, although they have been introduced independently of each other and were motivated by quite different considerations.

So one may select any one of these concepts for further study and be sure that the results obtained will have suitable analogues in the other theories. It becomes then largely a matter of choice which approach to choose. However, S-categories and pullback stripping functors, the latter more so, seem to be defined on an ad hoc basis. Their formulation is essentially a translation into categorical language of the properties which one thinks are basic. Top categories are different in the sense that they fit into a theory which is important in its own right: they are fibered categories in the sense of GROTHENDIECK [62] and GRAY [59]. In fact, top categories were earlier called T-fibered categories. Many results for top categories, as WYLER [172] observes, hold in general fibered categories as well. Also, top categories easily lend themselves to the study of topological algebra, in fact help in a slight generalization. For these reasons we think that aesthetically and mathematically top categories are more suitable for an exploitation of categorical ideas in general topology. Accordingly, the following three chapters are devoted to top categories.

CHAPTER - II

TOP CATEGORIES

In this chapter, we collect some general information about top categories. Section one lists some examples. Section two examines the general behaviour of a top category $\mathcal{A}^p$ with respect to $\mathcal{A}$. Properties such as constant-generation have been considered. It also explains some notational conventions and contains some preliminary results which are given in WYLER [172]. Section three examines the question of factorization of morphisms in $\mathcal{A}^p$ with respect to the same question in $\mathcal{A}$. It also exhibits the relationship between an equalizer and an extremal monomorphism.

2.1 Examples of top categories

A. Let $\mathcal{A}$ be any category. If pA is always the trivial lattice for any $A \in \mathcal{A}$ and α^p: $pA' \to pA$ is the unique map for any α: $A \xrightarrow{o} A'$ in $\mathcal{A}$, then $P: \mathcal{A}^p \to \mathcal{A}$ is an isomorphism of categories. Thus every category is a top category over itself with identity as the projection functor.

B. If A is a set, let pA be the lattice of all topologies on A where the order in pA is given by setting $x_1 \leqslant x_2$ iff x_2-open sets are also x_1-open. If for a function α: $A \to A'$ and a topology x' on A',

$\alpha^p x'$ denotes the topology on A given by the sets $\alpha^{-1}[V]$, $V \in x'$, then p: $ENS^{op} \to ORD$ is a topological theory. The top category ENS^p is easily seen to be TOP.

C. If GRP denotes the category of all groups and homomorphisms and pA, for any group A, stands for the set of all topologies on A which are compatible with the group structure, then proceeding exactly as in example B above, we get a topological theory p: $GRP^{op} \to ORD$. The top category GRP^p is the category of topological groups and continuous homomorphisms.

This example is typical of categories in topological algebra. WYLER [172] has given a general procedure. of constructing such examples.

D. If, for any set A, exp A denotes the set of all subsets of A, a <u>merotopy</u> on A is a set $X \subset$ exp exp A such that

1) If $x \in X$ and $x_1 \subset$ exp A such that to each $M \in x$, there exists an $M_1 \in x_1$ with $M_1 \supset M$, then $x_1 \in X$.
2) If $x \cup y \in X$ then either $x \in X$ or $y \in X$.
3) $((a)) \in X$ for each $a \in X$.

If mA denotes the set of all merotopies on A, define an order on mA by setting $X_1 \leqslant X_2$ iff $X_1 \subset X_2$. Then it

is easily seen that mA is a complete lattice. If $\alpha\colon A \to A'$ is a function and $X' \varepsilon\, mA'$ then let $\alpha^m X' = \sup\{X \varepsilon\, mA \mid \alpha X \subset X'\}$. Then $m\colon ENS^{op} \to ORD$ is a topological theory on ENS. The category ENS^m has objects (A,X) where X is a merotopy on A. Its morphisms $\alpha\colon (A,X) \to (A',X')$ are given by functions $\alpha\colon A \to A'$ such that $\alpha X \subset X'$. ENS^m is the category of <u>merotopic spaces</u> and <u>merotopically continuous functions</u>.

Merotopic spaces were defined by KATETOV [110]. Similar examples of top categories are the so called limit spaces, uniform limit space etc. (cf WYLER [172]).

E. If rA stands for the lattice of reflexive relations on a set A ordered by set inclusion and $\alpha\colon A \to A'$ is a function, let $\alpha^r\colon rA' \to rA$ be defined by $\alpha^r x' = \{(x_1,x_2) \mid (\alpha x_1, \alpha x_2)\, \varepsilon\, x'\}$. The top category ENS^r is the category of reflexive relations.

F. If, in example E above, rA is replaced by pA, the lattice of pre-orders on A, and we proceed as above, ENS^p is the category of pre-ordered sets and order-preserving functions. Similarly, if we start with eA, the lattice of equivalence relations on A, we get ENS^e, the category of equivalence relations.

LORRAIN [130] has proved that both these are full subcategories of TOP.

G. If A is a set, a <u>tolerance</u> on A is a reflexive and symmetric relation on A. Then if in example E above rA is replaced by $tl\,A$, the lattice of all tolerances on A and we proceed similarly, the resulting category ENS^{tl} is the category of <u>tolerance spaces</u> and <u>tolerance - continuous functions</u>.

This catogory, which may be denoted by TOL, is very useful in automata theory and was introduced there by ARBIB [104].

H. If A is a set, a <u>fuzzy set</u> in A is a function $x\colon A \to [0,1]$ where $[0,1]$ denotes the closed unit interval. The set of all fuzzy sets is ordered by setting $x_1 \leqslant x_2$ iff $x_1 a \leqslant x_2 a$ for all $a \in A$ and is then a complete lattice, say pA. For a function $a\colon A \to A'$ and a fuzzy set $x'\colon A' \to [0,1]$ let $a^p x' = x' a\colon A \to [0,1]$. Then $p\colon \mathrm{ENS}^{op} \to \mathrm{ORD}$ is a topological theory on ENS. The top category ENS^p has objects (A,x) where A is a set and x is a fuzzy set in A. A morphism $a\colon (A,x) \to (A',x')$ in ENS^p is given by a function $a\colon A \to A'$ such that $x \leqslant x' a$.

Fuzzy sets were introduced by ZADEH [177], to study situations in decision theory, pattern recognition etc. which are fuzzy rather than statistical. Since

then they have been found useful in several areas such as automata theory (MIZUMOTO, TOYODA and TANAKA [145]) and switching systems (MARINOS [138]). GOGUEN [56] generalized this notion to the so called V-sets by replacing $[0,1]$ by an arbitrary partly ordered set V. While V-sets will not form a top category in general, it is clear that $[0,1]$ can be replaced by any complete lattice L and the L-fuzzy sets as defined by GOGUEN [55] form a top category (but not with his morphisms). In particular, when $[0,1]$ is replaced by the two-element set $\{0,1\}$ with the order $0 < 1$, the resulting top category is the category of pairs of sets whose objects are (A,X) with X a subset of A, and morphisms $\alpha: (A,X) \to (A',X')$ are functions $\alpha: A \to A'$ such that $\alpha X \subset X'$

I. A fuzzy topology on a set A may be obtained by replacing sets by fuzzy sets in the usual definition of topology. Explicitly, a fuzzy topology on A is a set X of fuzzy sets in A such that

1) The constant functions $0, 1: A \to [0,1]$ are in X
2) If x_1, x_2 are in X then $\inf(x_1, x_2)$ is in X.
3) If $\{x_i \mid i \in I\} \subset X$, $\sup\{x_i \mid i \in I\}$ is in X.

Defining $X_1 \leqslant X_2$ iff $x \in X_2$ implies $x \in X_1$, we may order the set pA of all fuzzy topologies on A. Then

pA is a complete lattice. If $\alpha: A \to A'$ is a function and $X' \varepsilon pA'$ then $\alpha^p x' = \{x'\alpha | x' \varepsilon X'\}$ is a fuzzy topology on A. It is not hard to see that $p: ENS^{op} \to ORD$ is a topological theory on ENS. Then the top category ENS^p is the category of fuzzy topological spaces and fuzzy continuous functions in the sense of CHANG [27].

2.2 <u>Generalities</u>

Below we list certain propositions. Quite a few of them are in WYLER [172] though not necessarily in the same words or even listed as propositions in [172]. No arguments have been provided for these results and we include them simply for the sake of completeness.

<u>Proposition A</u> (WYLER [172]) <u>If $\mathcal{A}$ is a top category over $\mathcal{A}_o$, then the projection functor $P: \mathcal{A} \to \mathcal{A}_o$ has a left adjoint right inverse α (or α_p if the situation demands it) obtained by putting $\alpha A = (A, \alpha_A)$ for $A \varepsilon \mathcal{A}_o$ and $\alpha\alpha = \alpha: \alpha A \to \alpha A'$ for $\alpha: A \to A'$ in $\mathcal{A}_o$. Dually, P has a right adjoint right inverse ω (or ω_p) obtained similarly.</u>

For $\alpha: A \to A'$ in $\mathcal{A}$ and $x \varepsilon pA$, $\alpha_p x \varepsilon pA'$ may be defined by requiring $\alpha_p x \leqslant x'$ iff $x \leqslant \alpha^p x'$ for all $x' \varepsilon pA'$. As has been observed in 1.2, this defines a <u>map</u> $\alpha_p: pA \to pA'$.

Proposition B (WYLER [172]) The maps α_p define a covariant functor from $\mathcal{A}$ to ORD.

Proposition C (WYLER [172] If $\alpha : A \to A'$ and $\alpha' : A' \to A''$ are in $\mathcal{A}$ and if $x \in pA$ and $x'' \in pA''$, then the following three statements are equivalent.

1) $\alpha'\alpha : (A, x) \to (A'', x'')$ is in $\mathcal{A}^p$.

2) $\alpha : (A, x) \to (A', \alpha^p x'')$ is in $\mathcal{A}^p$

3) $\alpha' : (A', \alpha_p x) \to (A'', x'')$ is in $\mathcal{A}^p$.

Proposition D (WYLER [172]) The maps α_p actually define a topological theory on $\mathcal{A}^{op}$ and the top category constructed is isomorphic to $(\mathcal{A}^p)^{op}$.

This means that as long as $\mathcal{A}$ is not specialized, the theory is self-dual. We will normally not mention duals of any results in our discussion.

For monomorphisms etc., we have

Proposition E (WYLER [172]) A morphism (x, α, x') of $\mathcal{A}^p$ is a monomorphism (epimorphism) in $\mathcal{A}^p$ iff α is a monomorphism (epimorphism) in $\mathcal{A}$. Isomorphisms in $\mathcal{A}^p$ are morphisms $(\alpha^p x', \alpha, x')$ where α is an isomorphism in $\mathcal{A}$.

It is usual to call a category balanced if a morphism which is both a monomorphism and an epimorphism must be an isomorphism. If $\mathcal{A}^p$ is balanced then $\mathcal{A}$ is clearly so and if $\alpha\colon A \to A'$ is an isomorphism in $\mathcal{A}$, $\alpha^p x' = \omega_A$ for every $x' \in pA'$. In particular, $1_A^p\, x = 1_{pA}(x) = \omega_A$ for every $x \in pA$ which is not possible unless pA is trivial for each A. That is,

Proposition F Non trivial top categories are not balanced.

Since abelian categories are balanced,

Proposition G Non trivial top categories are non-abelian.

The following definitions are standard. If $\alpha_i\colon A_i \to A$, $i = 1,2$ are two monomorphisms in $\mathcal{A}$, one says $\alpha_1 \leqslant \alpha_2$ iff these exists a (necessarily unique monomorphism) $\alpha\colon A_1 \to A_2$ such that $\alpha_1 = \alpha_2\alpha$. With this ordering, the class of all monomorphisms with A as codomain is partly ordered in the following sense: if $\alpha_i\colon A_i \to A$, $i = 1,2$ are two monomorphisms such that $\alpha_1 \leqslant \alpha_2$ and $\alpha_2 \leqslant \alpha_1$ then there exists an isomorphism $\alpha\colon A_1 \to A_2$ such that $\alpha_2\alpha = \alpha_1$ and $\alpha_2 = \alpha_1\alpha^{-1}$. An equivalence class of monomorphisms with A as codomain is called a subobject of A. A category is locally small if every object has only a set of subobjects.

Quotient objects and colocally small categories are defined dually.

The following proposition is then obvious.

Proposition H $\mathcal{A}^p$ is locally (colocally) small iff $\mathcal{A}$ is.

The next result is due to WYLER [174]

Theorem I The following categories are not colocally small: separated limit spaces, separated neighbourhood spaces, separated uniform limit spaces and complete separated uniform limit spaces

Obviously then,

Corollary J The categories mentioned in theorem I above are not top categories (over any reasonable subcategory of EMS).

It follows that

Corollary K A sub category of a top category $\mathcal{A}^p$ need not be a top category over a subcategory of $\mathcal{A}$.

Of course, the conclusion in corollary K above is evident from the simpler facts that epimorphisms in T_o-spaces (cf. BARON [13]) and in Hausdorff spaces are not necessarily surjections.

For limits in $\mathcal{A}^p$ we have the following

Theorem L (WYLER [172]) A diagram $\Delta: \mathfrak{D} \to \mathcal{A}^p$ has a limit in $\mathcal{A}^p$ iff the diagram $P\Delta: \mathfrak{D} \to \mathcal{A}$ has a limit in $\mathcal{A}$.

In particular,

<u>Corollary M</u> <u>The initial object (the terminal object) in $\mathcal{A}^p$ is given by (A, α_A) $(..(A,\omega_A))$ where A is the initial object (the terminal object) in $\mathcal{A}$.</u>

If $\mathcal{A}$ has a zero object (an object which is both initial and terminal) then it does not follow that $\mathcal{A}^p$ has one. An example is given by taking $\mathcal{A}$ to be GRP, pA, for any group A, to be the closed interval $[0,1]$, α^p, for any homomorphism α, to be identity on $[0,1]$. Then a morphism $\alpha: (A,x) \to (A',x')$ in $\mathcal{A}^p$ is simply a homomorphism $\alpha: A \to A'$ such that $0 \leqslant x \leqslant x' \leqslant 1$. Obviously there is no zero object in $\mathcal{A}^p$ because the initial object is $(*,0)$ and the terminal object is $(*, 1)$ where $*$ is the group consisting of just one element. Therefore, additivity and preabelianness etc. do not automatically lift from $\mathcal{A}$ to $\mathcal{A}^p$. If one wants them to be lifted one has to assume that the lattice of structures on the zero object is trivial. Frequently (but not always-replace GRP by ENS in the example just constructed) the same is true for initial and terminal objects of $\mathcal{A}$. The case of the terminal object is a bit interesting and we discuss it in some detail.

If $\mathcal{A}$ has a terminal object, say T, then HERRLICH [77] defines a morphism $\alpha: A \to A'$ to be <u>constant</u> iff there is a factorization $A \to A' = A \to T \to A$. (Actually HERRLICH has a definition for general categories but we shall restrict ourselves to categories with a terminal object in which case the definition above is an equivalent formulation). If $\mathcal{A}$ is an <u>M-category</u> (HERRLICH [177] in the sense that $\mathcal{A}\ (A,A')$ is always non-empty then any $A \to T$ has a right inverse and the above $\alpha'': T \to A'$ is unique. Thus for an M-category with a terminal object T, it is possible to decide the constantness or otherwise of a morphism with clarity. If $\mathcal{A}^p$ is such a category, clearly so is $\mathcal{A}$. The converse is not true in general. Indeed, non-empty sets form an M-category (with a terminal object) and fuzzy sets in non-empty sets form a top category over them but there is obviously no morphism of fuzzy sets from the fuzzy set in A which takes A to 1 in [0,1] to the fuzzy set in A which takes A to 0 in [0,1]. The key to this situation is given by observing that fuzzy sets in the one-point set are more than one (in fact, a continuum). Precisely, we have the following

Proposition N Let $\mathcal{A}$ be an M-category with a terminal object T and let $\mathcal{A}^p$ be a top category over $\mathcal{A}$. Then $\mathcal{A}^p$ is an M-category iff pT is trivial.

Proof: Assume that $\mathcal{A}^p$ is an M-category. Then

$$1_T: (T,\omega_T) \to (T,\alpha_T) \text{ has to be in } \mathcal{A}^p$$

i.e. $\omega_T = \alpha_T$ and pT is trivial. Conversely, assume that pT is trivial. Then if (A,x) and (A',x') are any two objects in $\mathcal{A}^p$ and α: $T \to A'$ is in $\mathcal{A}$ then $(A,x) \to (T,pT) \to (A',x')$ has to be in $\mathcal{A}^p$, i.e. $\mathcal{A}^p$ is an M-category.

An M-category is constant-generated (HERRLICH [77]) iff for every pair of distinct morphisms α_1, α_2: $A \to A'$ there exists an object Z and a constant morphism k: $Z \to A$ with $\alpha_1 k \neq \alpha_2 k$. For those with a terminal object T, Z can always be taken to be T. Non-empty topological spaces are constant-generated. Some interesting results hold in constant-generated categories-for example every coreflective subcategory is both monocoreflective and epicoreflective (HERRLICH [77]).

If now $\mathcal{A}$ is constant-generated with a terminal object T and α: $(A,x) \to (A',x')$ is in $\mathcal{A}^p$ then α: $A \to A'$ is in $\mathcal{A}$ and factors as $A \overset{\alpha}{\to} A' = A \to T \overset{\alpha'}{\to} A'$ so that $(A,x) \overset{\alpha}{\to} (A',x') = (A,x) \to (T,pT) \overset{\alpha'}{\to} (A',x')$ is

a factorization in $\mathcal{A}^p$ provided that pT is trivial. We have then the following

Corollary 6 *In proposition N above, 'M-category' can be replaced by 'constant-generated'.*

2.3 Bicategories

In ENS, every function can be factorized as a surjection followed by an injection. This is an important property when considered in categorical setting. For example, in trying to characterize reflective subcategories of a category, one invariably looks for a suitable "factorization-through-the- image"-property of morphisms. The most pragmatic axiomatization is due to ISBELL [96]

Definition A Let $\mathcal{A}$ be a category. A *bicategory structure* on $\mathcal{A}$ is a pair (B_o, B_1) such that

BlC1 Every isomorphism is in $B_o \cap B_1$

BlC2 B_o and B_1 are closed under composition

BlC3 Every morphism α of $\mathcal{A}$ can be written as $\alpha = \alpha_1\alpha_o$ with $\alpha_i \varepsilon B_i$, $i = 0,1$. Furthermore, this factorization is unique in the sense that if $\alpha = \alpha_1' \alpha_o'$ then there is an isomorphism e for which $e\alpha_o = \alpha_o'$, $\alpha_1' e = \alpha_1$

B1C4 B_0 is a subclass of epimorphisms of $\mathcal{A}$

B1C5 B_1 is a subclass of monomorphisms of $\mathcal{A}$

<u>Proposition B</u> <u>If (B_0, B_1) is a bicategory structure on $\mathcal{A}$, $\mathcal{A}^p$ is a top category over $\mathcal{A}$, and</u>

$$B_0^p = \{\alpha\colon (A,x) \to (A',x') \mid \alpha \in B_0\}$$

$$B_1^p = \{\alpha\colon (A,\alpha^p x') \to (A',x') \mid \alpha \in B_1\},$$

<u>then (B_0^p, B_1^p) is a bicategory structure on $\mathcal{A}^p$.</u>

<u>Proof</u> An isomorphism in $\mathcal{A}^p$ is of the form $\alpha\colon (A,\alpha^p x') \to (A',x')$ with $\alpha\colon A \to A'$ an isomorphism in $\mathcal{A}$. Thus $\alpha \in B_0 \cap B_1$ and hence $(\alpha^p x', \alpha, x') \in B_0^p \cap B_1^p$.
If $\alpha\colon (A,x) \to (A',x')$ and $\alpha'\colon (A',x') \to (A'',x'')$ are in B_0^p then so is $\alpha'\alpha\colon (A,x) \to (A'',x'')$. Also, if $\alpha\colon (A,\alpha^p x')$ $\to (A',x')$ and $\alpha'\colon (A', \alpha'^p x'') \to (A'',x'')$ are in B_1^p and they are composable then $x' = \alpha'^p x''$ and hence $(x, \alpha'\alpha, x'')$ $= ((\alpha'\alpha)^p x'', \alpha'\alpha, x'')$ is in B_1^p. Finally, if $\alpha\colon (a,x) \to (A',x')$ is in $\mathcal{A}^p$ then there is a B_0-B_1-factorization $A \xrightarrow{\alpha} A' = A \xrightarrow{\alpha_0} A'' \xrightarrow{\alpha_1} A'$. Clearly, $(A,x) \xrightarrow{\alpha} (A',x') = (A,x) \xrightarrow{\alpha_0} (A'', \alpha_1^p x') \xrightarrow{\alpha_1} (A',x')$ is a B_0^p - B_1^p - factorization. If $(A, x) \xrightarrow{\alpha_0'} (A'', \alpha_1'^p x')$ $\to (A',x')$ is another B_0^p - B_1^p - factorization then $\alpha_1' e = \alpha_1$ so that $\alpha_1^p x' = (\alpha_1' e)^p x' = e^p \alpha_1'^p x'$ and

$(a_1^p x', e, a_1'^p x')$ is an $\mathcal{A}^p$-isomorphism. BIC1,2,3 have now been proved to be satisfied. BIC4 and 5 are obvious.

It is not hard to prove (cf. KANNISON [114] that if (B_0, B_1) is a bicategory structure on $\mathcal{A}$ then every diagram

$$\begin{tikzcd} \cdot \arrow[r, "a_0 \varepsilon B_0"] \arrow[d] & \cdot \arrow[d] \arrow[dl, dashed] \\ \cdot \arrow[r, "a_1 \varepsilon B_1"'] & \cdot \end{tikzcd}$$

can be filled in uniquely at the dotted arrow so as to render everything commutative--the commutativity of the outer sequare, of course, being a hypothesis. When B_0 is the class of all epimorphisms, monomorphisms satisfying this "diagram-diagonalization property" have been called <u>strong monomorphisms</u> (KELLEY [111]), <u>strict monomorphisms</u> (JURCHESCU and LASCU [103]) or just plain <u>monomorphisms</u> (ARDUINI [2]). They seem to be most useful in categories in general topology and functional analysis. A slightly weaker, but nonetheless very useful, concept is that of an <u>extremal monomorphism</u> defined by ISBELL [96]. These are monomorphisms m such that $m = fe$ and e an epimorphism implies that e is an isomorphism. Clearly,

<u>Proposition D</u> <u>Extremal monomorphisms in $\mathcal{A}^p$ are of the form $(m^p x', m_* x')$ with m an extremal monomorphism in $\mathcal{A}$.</u>

The following results have been obtained by several persons. A convenient reference is HERRLICH [73]

Proposition E Extremal monomorphisms are closed under composition iff they are strong monomorphisms

Proposition F The composition of an extremal monomorphism and a coretraction is again an extremal monomorphism

Proposition G Every equalizer is an extremal monomorphism.

JURCHESCU and LASCU [103] show that the proposition G above can be strengthened to read: every equalizer is a strong monomorphism. Below we prove a result which goes in the converse direction of proposition G.

Consider the following diagram

$$\begin{tikzcd} A \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & B \arrow[r, "f"] & C \end{tikzcd}$$

We say that (f_1, f_2) is a kernel pair of f if $ff_1 = ff_2$ and whenever $f'_i: A' \to B$, $i = 1,2$, are two morphisms such that there exists a unique morphism $\bar{f}: A' \to A$ such that $\bar{f}f_i = f'_i$. If (f'_1, f'_2) is also a kernel pair of f then there exists $\bar{f}' : A \to A'$ such that $f'_i\bar{f}' = f_i$. Then by uniqueness requirements $\bar{f}\bar{f}' = 1_A$ and $\bar{f}'\bar{f} = 1_{A'}$.

That is, upto an isomorphism, kernel pairs are unique. The diagram above may be said to be <u>exact</u> if (f_1, f_2) is a kernel pair of f and f is a coequalizer of (f_1, f_2). We may abuse the language and say that f is an <u>exact</u> coequalizer of (f_1, f_2). Dually, one has exact equalizers.

<u>Theorem H</u> <u>Let $\mathcal{A}$ be a category such that if $f_i: A \to B$, $i = 1,2$ are extremal monomorphisms then their coequalizer always exists and is exact. Then if $\mathcal{A}$ has finite coproducts, every extremal monomorphism in $\mathcal{A}$ is an exact equalizer.</u>

<u>Proof</u>: The argument is summarized in the diagram below

$$\begin{tikzcd}
 & & B_1 \arrow[d, "\beta_1"] & & \\
A \arrow[r, "m"] & B \arrow[ur, "b_1"] \arrow[dr, "b_2"'] \arrow[rrr, bend left=50, "\beta'_1"] \arrow[rrr, bend right=50, "\beta'_2"'] & B_1+B_2 \arrow[l, "b"'] \arrow[r, "e"] & C & B' \arrow[l, dashed, "k"'] \\
A' \arrow[u] \arrow[ur, "h"'] & & B_2 \arrow[u, "\beta_2"'] & &
\end{tikzcd}$$

Explicitly, let $m: A \to B$ be an extremal monomorphism. Let B_1 and B_2 be two isomorphic copies of B with $b_i: B \to B_i$, $i = 1,2$ as the corresponding isomorphisms. Let $B_1 + B_2$ be the coproduct of B_1 and B_2 with β_i, $i = 1,2$ as canonical injections. Then the isomorphisms

$b_i^{-1}: B_i \to B$ induce a morphism $b: B_1 + B_2 \to B$ such that $b\beta_i = b_i^{-1}$. Thus both $\beta_1 b_1$ and $\beta_2 b_2$ are coretractions so that the compositions $\beta_i b_i m$, $i = 1,2$ are extremal monomorphisms. Let $e: B_1 + B_2 \to C$ be the coequalizer of $\beta_i b_i m$, $i = 1,2$. By assumption, this is exact. If now $h: A' \to B$ is such that $(e\,\beta_1 b_1)h = (e\,\beta_2 b_2)h$ then since $(\beta_1 b_1 m, \beta_2 b_2 m)$ is the kernel of e, there exists a unique $\bar{h}: A' \to A$ such that $\beta_i b_i m\bar{h} = \beta_i b_i h$. Since $\beta_i b_i$ are coratractions, $m\bar{h} = h$. The uniqueness of $\bar{h}$ is obvious and m is the equalizer of $e\beta_1 b_1$ and $e\beta_2 b_2$. If now $\beta'_i : B \to B'$, $i = 1,2$ is another pair of morphisms with $\beta'_1\, m = \beta'_2\, m$ then the morphisms $\beta'_i\, b_i^{-1} : B_i \to B'$ induce a unique morphism $\beta: B_1 + B_2 \to B'$ (not shown in the diagram) such that $\beta_i b_i^{-1} = \beta\beta_i$. Then $\beta\beta_1 b_1 m = \beta\beta_2 b_2 m$. Since e is the equalizer of $\beta_1 b_1 m$ and $\beta_2 b_2 m$ there exists a unique morphism $k: C \to B'$ such that $ke = \beta$. Then $k\, e\, \beta_i b_i = \beta\beta_i b_i = \beta'_i$. The uniqueness of k to ascertain that $(e\beta_1 b_1, e\beta_2 b_2)$ is indeed the 'cokernel pair' of m is easy to verify. Thus m is the exact equalizer of $e\beta_1 b_1$ and $e\beta_2 b_2$.

When $\mathcal{A}^P$ is a top category over $\mathcal{A}$,

<u>Theorem I</u> <u>Let $\mathcal{A}$ be as in theorem H above. Then every extremal monomorphism in $\mathcal{A}^P$ is an exact equalizer.</u>

Proof: One proceeds exactly as above. Remembering that P preserves equalizers, coming via P to $\mathcal{A}$ and going back again, the desired conclusion is obtained.

Theorem I improves theorem 1 of HERRLICH and STRECKER [80] which says that in TOP, topological embeddings, equalizers and extremal monomorphisms are identical.

CHAPTER - III

LIFTING FUNCTORS

In this chapter, we discuss the behaviour of functors $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ which 'lift' functors $F: \mathcal{A} \to \mathcal{B}$. Section one is a resume of some definitions and results from [172]. Section two gives a theorem which says that if $\mathcal{A}$ is small, diagrams in $\mathcal{B}^q$ over $\mathcal{A}^p$, which lift diagrams in $\mathcal{B}$ over $\mathcal{A}$, form a top category over them, and uses this to derive a result of WYLER [172]. It is quite natural to ask what properties of functors are invarient under lifting and section three provides the answer for some basic properties. Section four proves that the subequalizer of a pair of functors Φ and Ψ which lift F and G respectively, is a top category over the subequalizer of F and G. As corollaries, it observes that the comma category of Φ and Ψ is a top category over that of F and G and gives a simple proof of a result of WYLER [172] concerning lifting of universal morphisms. Section five studies lifting of local adjuction. Finally, section six gives an example which solves a problem raised by WYLER.

3.1 Definitions and basic information

If $\mathcal{A}^p$ is a top category with the projection functor $P: \mathcal{A}^p \to \mathcal{A}$ and $\mathcal{B}^q$ is a top category with the

projection functor $Q: \mathcal{B}^q \to \mathcal{B}$ then we say that a functor $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ lifts a functor $F: \mathcal{A} \to \mathcal{B}$ if $Q\Phi = FP$. If Φ and Ψ lift F and G respectively then we say that a natural transformation $\Lambda: \Phi \to \Psi$ lifts a natural transformation $\lambda: F \to G$ if $Q\Lambda = \lambda P$. If Φ lifts F then $F = Q\Phi\, \alpha_p = Q\Phi\, \omega_p$ and thus Φ determines F.

Proposition A (WYLER [172]) Let $F: \mathcal{A} \to \mathcal{B}$ be a functor. If maps $\varphi_A: pA \to qFA$ in ORD are given, one for every object A of $\mathcal{A}$, and if

1) $\varphi_A a^p x' \leqslant (Fa)^q \varphi_{A'} x'$

in qFA whenever $a: A \to A'$ is in $\mathcal{A}$ and $x' \in pA'$, then

2) $\Phi(A,x) = (FA, \varphi_A x)$, $\Phi a = Fa: \Phi(A,x) \to \Phi(A',x')$

for objects (A,x) and morphisms $a: (A,x) \to (A',x')$ of $\mathcal{A}^p$, defines a functor $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ which lifts F. Every functor $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ which lifts F is obtained in this way.

The maps φ_A are called the structure maps of the functor Φ. If Φ lifts F each φ_A preserve infima and 1) is an equality, we say that Φ is taut over F.

Proposition B (WYLER [172]) $1_{\mathcal{A}^p}$ lifts $1_{\mathcal{A}}$, with structure maps 1_{pA}. If $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ and $\Psi: \mathcal{B}^q \to \mathcal{C}^r$

lift $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{C}$ with structure maps φ_A and ψ_B respectively then $\Psi\Phi$ lifts GF, with structure maps $\psi_{FA}\varphi_A$.

Proposition C (WYLER [172]) If functors Φ and Ψ from $\mathcal{A}^p$ to $\mathcal{B}^q$ lift functors F and G from $\mathcal{A}$ to $\mathcal{B}$, with structure maps φ_A and ψ_A, then every natural transformation $\Lambda: \Phi \to \Psi$ lifts a natural transformation $\lambda: F \to G$, and in this situation Λ and λ determine each other.

Proposition D (WYLER [172]) Λ in proposition C above is a natural equivalence iff λ is a natural equivalence and $\varphi_A = (\lambda A)^q \psi_A$ for every $A \in \mathcal{A}_o$.

3.2 Lifting of functors is a topological theory

In general, a functor $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ may not lift any functor $F: \mathcal{A} \to \mathcal{B}$. The following simple example, due to WYLER [172], illustrates this situation. Let $\mathcal{A}^p$ be the category of pairs of sets (cf. example 2.1H) and let $\mathcal{B}^q = ENS$, both considered as top categories over ENS. Put $T(A,X) = X$ for a pair of sets and let $T\alpha: X \to X'$ be the restriction of α for $\alpha: (A,X) \to (A',X')$. Since $P(A,X) = A$ and $QX = X$ in this situation, there can be no functor F: ENS $\to$ ENS such that $QT = FP$. On the other hand, usually many functors $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ will lift a given $F: \mathcal{A} \to \mathcal{B}$ (two obvious ones are

$\alpha_q FP$ and $\omega_q FP$). In view of this, it is of some interest to prove the following

<u>Theorem A</u> <u>Let $\mathcal{A}^p$ and $\mathcal{B}^q$ be top categories. If $\mathcal{A}$ is small, then the functors $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ which lift functors $F: \mathcal{A} \to \mathcal{B}$ form a top category over them.</u>

<u>Proof</u> Since $\mathcal{A}$ is small, there is at most a set of structure maps $\varphi_A: pA \to qFA$, and consequently of functors $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ which lift a given $F: \mathcal{A} \to \mathcal{B}$. Denote this set by rF and define an order as follows: if Φ^1, Φ^2 lift F then $\Phi^1 \leqslant \Phi^2$ iff $\varphi^1_A \leqslant \varphi^2_A$ for every $A \in \mathcal{A}_o$ in the sense that $\varphi^1_A x \leqslant \varphi^2_A x$ in qFA for every $x \in pA$. If $\{\Phi^\iota \mid \iota \in I\}$ is a set of functors lifting F, $\Phi = \sup \Phi^\iota$ is defined to have structure maps φ_A such that $\varphi_A x = \sup \varphi^\iota_A x$. If $\alpha: A \to A'$ is in $\mathcal{A}$ and $x' \in pA'$ then

$$\varphi^\iota_A \ \alpha^p x' \leqslant (F\alpha)^q \ \varphi^\iota_{A'} \ x'$$

so that taking sup on both sides of this inequality, $\varphi_A \alpha^p x' \leqslant (F\alpha)^q \varphi_{A'} x'$; that is, φ_A are structure maps indeed and Φ is a well defined functor lifting F. Similarly $\inf \Phi^\iota$ is also defined. Thus rF is a complete lattice. If $G: \mathcal{A} \to \mathcal{B}$ is another functor and $\lambda: F \to G$ is a natural transformation then for every $A \in \mathcal{A}_o$, there is a morphism $\lambda A: FA \to GA$ such that $(G\alpha)\lambda A = (\lambda A')F\alpha$

for every $\alpha: A \to A'$ in $\mathcal{A}$. Assuming now that $\Psi: \mathcal{A}^p \to \mathcal{B}^q$ lifts G, we construct a functor $\lambda^r\Psi: \mathcal{A}^p \to \mathcal{B}^q$ lifting F. It suffices to provide structure maps; we choose them to be

$$pA \xrightarrow{(\lambda^r\psi)_A} qFA = pA \xrightarrow{\psi_A} qGA \xrightarrow{(\lambda A)^q} qFA$$

If $\alpha: A \to A'$ is in $\mathcal{A}$ and $x' \varepsilon pA'$, then

$$\begin{aligned}(\lambda^r\psi)_A(\alpha^p x') &= (\lambda A)^q \psi_A \alpha^p x' \\ &\leqslant (\lambda A)^q (G\alpha)^q \psi_{A'} x' \quad \text{since } \Psi \ \varepsilon \ rG, \\ &\qquad\qquad (\lambda A)^q \varepsilon \ \mathrm{ORD} \\ &= (F\alpha)^q (\lambda A')^q \psi_{A'} x' \\ &= (F\alpha)^q (\lambda^r\psi)_{A'} x'\end{aligned}$$

so that $(\lambda^r\psi)_A$ are structure maps indeed and $\lambda^r\Psi \ \varepsilon \ rF$. If $\Psi^1 \leqslant \Psi^2$ so that $\psi^1_A \leqslant \psi^2_A$ for each $A \ \varepsilon \ \mathcal{A}_o$ then

$$(\lambda^r\psi^1)_A x = (\lambda A)^q \psi^1_A x \leqslant (\lambda A)^q \psi^2_A x = (\lambda^r\psi^2)_A x$$

for every $x \ \varepsilon \ pA$ since $(\lambda A)^q \ \varepsilon \ \mathrm{ORD}$. Thus each λ^r is order preserving. Again, if $\{\Psi^i \mid i \ \varepsilon \ I\} \subset rG$ and $\Psi = \inf \Psi^i$, then

$$\begin{aligned}(\lambda^r\psi)_A x = (\lambda A)^q \psi_A x &= (\lambda A)^q \inf(\psi^i_A x) \\ &= \inf((\lambda A)^q \psi^i_A)x \text{ so that}\end{aligned}$$

$\lambda^r\Psi = \inf(\lambda^r\Psi^i)$ and each λ^r preserves infima.

Thus r is a topological theory on the functor category $(\mathcal{A}, \mathcal{B})$. The top category $(\mathcal{A},\mathcal{B})^r$ has objects (F,Φ) where $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ lifts $F: \mathcal{A} \to \mathcal{B}$. A morphism (Φ,λ,Ψ): $(F,\Phi) \to (G,\Psi)$ is given by a natural transformation $\lambda: F \to G$ such that $\Phi \leqslant \lambda^r\Psi$. Such a morphism defines, for each $(A,x) \in \mathcal{A}^p_o$, a morphism $\Lambda(A,x) = (\varphi_A x, \lambda A, \psi_A x): \Phi(A,x) \to \Psi(A,x)$ in $\mathcal{B}^q$ such that if $\alpha: (A,x) \to (A',x')$ is a morphism in $\mathcal{A}^p$, then the following diagram

$$\begin{tikzcd}
(FA, \varphi_A x) \arrow[r, "\lambda A"] \arrow[d, "F\alpha"'] & (GA, \psi_A x) \arrow[d, "G\alpha"] \\
(GA, \varphi_{A'} x') \arrow[r, "\lambda A'"'] & (GA', \psi_{A'} x')
\end{tikzcd}$$

commutes. In other words, this determines a natural transformation $\Lambda: \Phi \to \Psi$ which lifts λ. Conversely, if $\Lambda: \Phi \to \Psi$ lifts $\lambda: F \to G$, then for any $(A,x) \in \mathcal{A}_o$ $\Lambda(A,x) = (\varphi_A x, \lambda A, \psi_A x)$ is in $\mathcal{B}^q$ which means that $\varphi_A x \leqslant (\lambda A)^q \psi_A x$ i.e. $\Phi \leqslant \lambda^r \Psi$. Identifying (F,Φ) to Φ and (Φ, λ, Ψ) to the corresponding $\Lambda: \Phi \to \Psi$ the theorem follows.

This theorem has been included in [172].

In particular, when each pA is the trivial lattice so that $\mathcal{A}^p = \mathcal{A}$, diagrams in $\mathcal{B}^q$ over $\mathcal{A}$ are a top

category over diagrams in $\mathcal{B}$ over $\mathcal{A}$. Indeed, in this case it is easily seen that a diagram $\Phi: \mathcal{A} \to \mathcal{B}^q$ lifting a diagram $F: \mathcal{A} \to \mathcal{B}$ is characterised by a family $\{y_A \mid A \in \mathcal{A}_o\}$ with $y_A \in qFA$ for every $A \in \mathcal{A}_o$ such that $y_A \leqslant (F\alpha)^q y_{A'}$ for any morphism $\alpha: A \to A'$ in $\mathcal{A}$. Also the order $\Phi^1 \leqslant \Phi^2$ is now given by $\{y^1_A \mid A \in \mathcal{A}_o\} \leqslant \{y^2_A \mid A \in \mathcal{A}_o\}$ iff $y^1_A \leqslant y^2_A$ in qFA for all $A \in \mathcal{A}_o$. These observations can be summarized in the form of a corollary which is actually theorem 3.8 in [172]. The corollary is evident; the remarks above show that the proof in [172] is along the same lines as the proof of theorem A.

<u>Corollary B</u> <u>The category of diagram in $\mathcal{B}^q$ is a top category over the category of diagram (over the same scheme) in $\mathcal{B}$.</u>

3.3 <u>Invariance under lifting</u>

We now look at the following question: Suppose $F: \mathcal{A} \to \mathcal{B}$ has a certain property. If $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ lifts F, does Φ also have the same property? The answer of course, is not always yes. For example, if $F: \mathcal{A} \to \mathcal{B}$ is the identity functor, then clearly $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ is not going to be identity just because it lifts F. The following proposition explain the situation about some important properties.

Proposition A. *Let $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ lift $F: \mathcal{A} \to \mathcal{B}$. Then Φ is faithful iff F is faithful. If Φ is taut over F and if each φ_A is full as a functor then Φ is full iff F is. Finally, Φ is representative iff each F is representative and each φ_A is onto.*

Proof: First, note that if lattices are regarded as categories and order preserving functions between them as functors, then a function F is representative iff it is onto and is full iff it is one-one and reflects the order in the sense that $fx \leqslant fy$ implies $x \leqslant y$. Let now $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ lift $F: \mathcal{A} \to \mathcal{B}$. If F is faithful and $\alpha_1, \alpha_2: (A,x) \to (A',x')$ are in $\mathcal{A}^p$ such that $\Phi(x,\alpha_1,x') = \Phi(x,\alpha_2,x')$ i.e. $F\alpha_1 = F\alpha_2: (FA,\varphi_A x) \to (FA', \varphi_{A'} x')$ is in $\mathcal{B}^q$, then $F\alpha_1 = F\alpha_2: FA \to FA'$ is in $\mathcal{B}$ so that $\alpha_1 = \alpha_2: A \to A'$ and hence $(x,\alpha_1,x') = (x,\alpha_2,x')$, i.e. Φ is faithful. Conversely, let Φ be faithful and $\alpha_1,\alpha_2: A \to A'$ be in $\mathcal{A}$ such that $F\alpha_1 = F\alpha_2$. Then $\alpha_1,\alpha_2: (A,\alpha_A) \to (A', \alpha_{A'})$ are in $\mathcal{A}^p$ and hence $F\alpha_1 = F\alpha_2: (FA, \varphi_A\alpha_A) \to (FA', \varphi_A \alpha_{A'})$ is in $\mathcal{B}^q$. Since Φ is faithful, $(\alpha_A,\alpha_1,\alpha_{A'}) = (\alpha_A,\alpha_2,\alpha_{A'})$ i.e. $\alpha_1 = \alpha_2$. Next, let F be full and let $b: (FA,\varphi_A x) \to (FA', \varphi_{A'} x')$ be in $\mathcal{B}^q$, then $b: FA \to FA'$ is in $\mathcal{B}$ and hence there is $\alpha: A \to A'$ such that $F\alpha = b$. If now Φ is taut over F then $\varphi_A x \leqslant b^q\varphi_{A'}x' = (F\alpha)^q\varphi_{A'}x' =$

$\varphi_A \alpha^p x'$ from which we can conclude that $x \leqslant \alpha^p x'$ if φ_A is full. Then (x, α, x') is in $\mathcal{A}^p$ with $\Phi(x,\alpha,x') = (\varphi_A x, b, \varphi_{A'} x')$ and Φ is full. Conversely, if Φ is full and $b: FA \to FA'$ is in $\mathcal{B}$ then $b: (FA, \alpha_{FA}) \to (FA', \alpha_{FA'})$ is in $\mathcal{B}^q$ and if Φ is taut over F then this is $b: (FA, \varphi_A \alpha_A) \to (FA', \varphi_{A'} \alpha_{A'})$. But $(\varphi_A \alpha_A, b, \varphi_{A'} \alpha_{A'})$ in $\mathcal{B}^q$ implies that $(\alpha_A, \alpha, \alpha_A): (A, \alpha_A) \to (A', \alpha_{A'})$ in $\mathcal{A}^p$ such that $\Phi(\alpha_A, \alpha, \alpha_{A'}) = (\varphi_A \alpha_A, b, \varphi_A \alpha_{A'})$ i.e. there is $\alpha: A \to A'$ in $\mathcal{A}$ with $F\alpha = b$. Thus F is full. Finally let F be representative. Then every object of $\mathcal{B}$ is (isomorphic to) FA for some $A \in \mathcal{A}_o$. If $(FA, y) \in \mathcal{B}^q_o$ and each $\varphi_A: pA \to qFA$ is onto, then clearly $(FA, y) = (FA, \varphi_A x)$ for some $x \in pA$ and Φ is representative. Conversely, let Φ be representative. Then every object of $\mathcal{B}^q$ is (isomorphic to) $(FA, \varphi_A x)$ for some $(A, x) \in \mathcal{A}^p_o$, i.e. F is representative and each φ_A is onto.

A little reflection proves the following very desirable

<u>Corollary B</u> <u>Let</u> $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ <u>lift</u> $F: \mathcal{A} \to \mathcal{B}$. <u>Then</u> Φ <u>is an equivalence iff</u> F <u>is an equivalence and</u> pA <u>is isomorphic to</u> $q\,FA$ <u>for every</u> $A \in \mathcal{A}_o$

3.4 Subequalizers, comma categories, universal morphisms

Let $F, G: \mathcal{A} \to \mathcal{B}$ be two functors. The subequalizer (LAMBEK [123]) of F and G is a triple $(\mathcal{S}, D, t)$ where $\mathcal{S}$ is a category, $D: \mathcal{S} \to \mathcal{A}$ is a functor and $t: FD \to GD$ is a natural transformation defined as follows. The objects of $\mathcal{S}$ are pairs (A,b) where $A \underset{o}{\in} \mathcal{A}$ and $b: FA \to GA$ is in $\mathcal{B}$. $(b,\alpha,b'): (A,b) \to (A',b')$ (or more simply, $\alpha: (A,b) \to (A',b')$ of $\mathcal{S}$ is given by a morphism $\alpha: A \to A'$ of $\mathcal{A}$ such that the following diagram commutes.

i)
$$\begin{tikzcd}
FA \arrow[r, "F\alpha"] \arrow[d, "b"'] & FA' \arrow[d, "b'"] \\
GA \arrow[r, "G\alpha"'] & GA'
\end{tikzcd}$$

D is given by $D(A,b) = A$ and $D(b,\alpha,b') = \alpha$, and t is given by $t(A,b) = b$. The subequalizer $(\mathcal{S}, D, t)$ has this universal property: whenever there is a triple $(\mathcal{S}', D', t')$ such that $D': \mathcal{S}' \to \mathcal{A}$ and $t': FD' \to GD'$, then there exists a unique functor $E: \mathcal{S}' \to \mathcal{S}$ such that $DE = D'$ and $tE' = t'$ $\mathcal{S}$ is called the <u>subequalizing category</u> of F and G.

Let us now assume that $\mathcal{A}^p$ and $\mathcal{B}^q$ are top categories with projections $P: \mathcal{A}^p \to \mathcal{A}$ and $Q: \mathcal{B}^q \to \mathcal{B}$ and Φ, Ψ: $\mathcal{A}^p \to \mathcal{B}^q$ are two functors such that Φ lifts $F: \mathcal{A} \to \mathcal{B}$ and Ψ lifts $G: \mathcal{A} \to \mathcal{B}$. If $(\mathcal{S}_1, D_1, t_1)$ is the subequalizer of Φ and Ψ then

1) Objects of $\mathcal{S}_1$ are pairs $(A,x),b)$ where $(A,x) \in \mathcal{A}^p$ and $b: \Phi(A,x) \to \Psi(A,x)$ i.e. $b: (FA, \varphi_A x) \to (GA, \psi_A x)$ is in $\mathcal{B}^q$.

2) A morphism α: $((A,x),b) \to ((A',x'),b')$ is in $\mathcal{S}_1$ iff α: $(A,x) \to (A',x')$ is in $\mathcal{A}^p$ and the following diagram commutes.

ii)
$$\begin{tikzcd}
(FA, \varphi_A x) \arrow[r, "F\alpha"] \arrow[d, "b"'] & (FA', \varphi_{A'} x') \arrow[d, "b'"] \\
(GA, \psi_A x) \arrow[r, "G_\alpha"'] & (GA', \psi_{A'} . x')
\end{tikzcd}$$

$D_1: \mathcal{S}_1 \to \mathcal{A}^p$ is given by $D_1((A,x),b) = (A,x)$ and $D_1(b,(x,\alpha,x'),b') = (x,\alpha,x')$, and $t_1: \Phi D_1 \to \Psi D_1$ is given by $t((A,x),b) = b$. We now assume that Ψ is taut over G and construct a top category $\mathcal{S}^r$ as follows. if $(A,b) \in \mathcal{S}_0$, define $r(A,b) = \{x \in pA | b: (FA, \varphi_A x) \to (GA, \psi_A x)$ is in $\mathcal{B}^q\}$. This is clearly a subset of pA. If now $\{x_i | i \in I\} \subset r(A,b)$ and x denotes inf x_i in pA then we have $\varphi_A x \leqslant \inf \varphi_A x_i \leqslant \inf b^q \psi_A x_i = b^q(\inf \psi_A x_i) = b^q \psi_A \inf x_i = b^q \psi_A x$ i.e. $b: (FA, \varphi_A x) \to (GA, \psi_A x)$ is in $\mathcal{B}^q$ and $x \in r(A,b)$. Thus $r(A,b)$ is a

sublattice of a complete lattice and is closed under infima so that it is a complete lattice (but not necessarily a complete sublattice). If $\{x_i \mid i \in I\} \subseteq r(A,b)$ then $\sup x_i$ is the inf in pA of those upper bounds in pA which belong to $r(A,b)$. In case there are no such upper bounds, the required sup is ω^p_A. To see that $\omega^p_A \in r(A,b)$, note that $b^q \psi_A \omega^p_A = b^q \omega^q_{GA} = \omega^q_{FA}$ (since both $\psi_A: pA \to qGA$ and $b^q: qGA \to qFA$ preserve infima) and hence $b: (FA, \varphi_A\omega^p_A) \to (GA, \psi_A\omega^p_A)$ is in $\mathscr{B}^q$ i.e., $\omega^p_A \in r(A,b)$. Thus, to every $(A,b) \in \overset{\circ}{\mathscr{A}}$ we have associated a complete lattice $r(A,b)$. If $\alpha: (A,b) \to (A',b')$ is in $\mathscr{A}$ and $x' \in r(A',b')$, we define $\alpha^r x' = \alpha^p x'$. To see that $\alpha^r x' \in r(A,b)$, note that since Φ lifts F, $\alpha: A \to A'$ is in $\mathscr{A}$ and $x' \in pA'$,

$$\begin{aligned}
\varphi_A \alpha^p x' &\leqslant (F\alpha)^q \varphi_{A'} x' \\
&\leqslant (F\alpha)^q b'^q \psi_{A'} x', \text{ since } x' \in r(A',b') \\
&= b^q (G\alpha)^q \psi_{A'} x' \text{ since (i) commutes} \\
&= b^q \psi_A \alpha^p x' \text{ since } \Psi \text{ is taut over } G
\end{aligned}$$

which says that $b: (FA, \varphi_A \alpha^p x') \to (GA, \psi_A \alpha^p x')$ is in $\mathscr{B}^q$, i.e. that $\alpha^r x' \in r(A,b)$. Thus for any given $\alpha: (A,b) \to (A',b')$ in $\mathscr{A}$ we have associated a function $\alpha^r: r(A',b') \to r(A,b)$. Since α^p is in ORD, so is α^r,

since a^p preserves infima, so does a^r ($r(A',b')$ has the same infima as pA') and thus $r\colon \mathcal{B}^{op} \to \text{ORD}$ is a topological theory on $\mathcal{B}$. The top category $\mathcal{B}^r$ with the projection functor $R\colon \mathcal{B}^r \to \mathcal{B}$, has objects $((A,b),x)$ where $(A,b) \in \mathcal{B}_o$ and $x \in r(A,b)$, i.e. $b\colon (FA, \varphi_A x) \to (GA, \psi_A x)$ is in $\mathcal{B}^q$. This means that $((A,b),x) \in \mathcal{B}^r_o$ iff $((A,x),b) \in \mathcal{B}_{1o}$. A morphism $a\colon ((A,b),x) \to ((A',b'),x')$ in $\mathcal{B}^r$ is given by a morphism $a\colon (A,b) \to (A',b')$ of $\mathcal{B}$ such that $x \leqslant a^r x'$, i.e. by a morphism $a\colon A \to A'$ of $\mathcal{A}$ such that (i) commutes and $x \leqslant a^r x' = a^p x'$. On the other hand a morphism $a\colon (A,x),b) \to ((A',x')b')$ of $\mathcal{B}_1$ is given by a morphism $a\colon (A,x) \to (A',x')$ of $\mathcal{A}^p$ such that (ii) commutes, i.e. by a morphism $a\colon A \to A'$ of $\mathcal{A}$ such that $x \leqslant a^p x' = a^r x'$ and (ii) commutes. A comparison of the definitions shows that $a\colon ((A,b),x) \to ((A',b'),x')$ is in $\mathcal{B}^r$ iff $a\colon ((A,x),b) \to ((A',x'),b')$ is in $\mathcal{B}_1$. Making the obvious identifications, we see that the subequalizing category of Φ and Ψ is a top category over the subequalizing category of F and G. Next, $PD_1((A,b),x)$

$$= PD_1((A,x),b) = P(A,x) = A = D(A,b) = DR((A,b),x)$$

so that $PD_1 = DR$ and D_1 lifts D. Moreover, the structure maps, say $\partial(A,b)$, of D_1 are just the inclusions

$r(A,b) \subset pA$, so that if $\alpha: (A,b) \to (A',b')$ is in $\mathcal{S}$ and $x' \varepsilon r(A',b')$, then $\partial_{(A,b)} \alpha^r x' = \alpha^p x' = (D\alpha)^p = \partial_{(A',b')} x'$, i.e. D_1 is actually taut over D. Finally, $Pt_1((A,b),x) = Pt_1((A,x),b) = P(\varphi_A x, b, \psi_A x) = b = t(A,b) = tR((A,b),x)$ so that $Pt_1 = tR$ and t_1 lifts t.

Summarizing, we have the following

Theorem A Let $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ lift $F: \mathcal{A} \to \mathcal{B}$ and let $\Psi: \mathcal{A}^p \to \mathcal{B}^q$ be taut over $G: \mathcal{A} \to \mathcal{B}$. Then if $(\mathcal{S}_1, D_1, t_1)$ is the subequalizer of Φ and Ψ, and $(\mathcal{S}, D, t)$ is the subequalizer of F and G, $\mathcal{S}_1$ is a top category over $\mathcal{S}$, D_1 lifts D tautly and t_1 lifts t.

As an application of theorem A above , we make the following observations. First if, $F_o: \mathcal{A}_o \to \mathcal{B}$, $F_1: \mathcal{A}_1 \to \mathcal{B}$ is a pair of functors and $\mathcal{A}_o \times \mathcal{A}_1$ is the product category with canonical projections $\pi_i: \mathcal{A}_o \times \mathcal{A}_1 \to \mathcal{A}_i$, $i = 0,1$ then the subequalizing category of $F_o\pi_o$ and $F_1\pi_1$ is precisely the comma category (F_o, F_1) in the sense of LAWVERE [125]. Also,

Lemma B If $\mathcal{A}_o^{p_o}$ and $\mathcal{A}_1^{p_1}$ are top categories with projection functors $P_i: \mathcal{A}_i^{p_i} \to \mathcal{A}_i$, $i = 0,1$ then the product category $\mathcal{A}_o^{p_o} \times \mathcal{A}_1^{p_1}$ is a top category over $\mathcal{A}_o \times \mathcal{A}_1$ with the projection functor $P = P_o \times P_1$.

<u>Proof</u>: For $(A_o, A_1) \underset{o}{\varepsilon} \mathcal{A}_o \times \mathcal{A}_1$ define $p(A_o, A_1) = p_o A_o \times p_1 A_1$. This is a complete lattice. If $(\alpha_o, \alpha_1): (A_o, A_1) \to (A'_o, A'_1)$ is in $\mathcal{A}_o \times \mathcal{A}_1$ and $(x'_o, x'_1) \in p(A'_o, A'_1)$, define $(\alpha_o, \alpha_1)^p: p(A'_o, A'_1) \to p(A_o, A_1)$ by setting $(\alpha_o, \alpha_1)^p (x'_o, x'_1) = (\alpha_o^{p_o} x'_o, \alpha_1^{p_1} x'_1)$. Then $p: (\mathcal{A}_o \times \mathcal{A}_1)^{op} \to ORD$ is a topological theory on $\mathcal{A}_o \times \mathcal{A}_1$ and making the obvious identifications, $(\mathcal{A}_o \times \mathcal{A}_1)^p$ is seen to be isomorphic to $\mathcal{A}_o^{p_o} \times \mathcal{A}_1^{p_1}$.

If now $\Pi_i: \mathcal{A}_o^{p_o} \times \mathcal{A}_1^{p_1} \to \mathcal{A}_i^{p_i}$, $i = 0,1$ denote the canonical projections then

$$P_i \Pi_i ((A_o, A_1), (x_o, x_1)) = P_i \Pi_i ((A_o, x_o), (A_1, x_1))$$

$$= P_i (A_i, x_i) = A_i = \pi_i (A_o, A_1) = \pi_i P ((A_o, A_1), (x_o, x_1))$$
$$i = 0,1$$

so that Π_i lift π_i. Moreover, this is a taut lifting. Consequently, if $\Phi_1: \mathcal{A}_1^p \to B^q$ is taut over $F_1: \mathcal{A}_1 \to \mathcal{B}$ then $\Phi_1 \Pi_1: (\mathcal{A}_o \times \mathcal{A}_1)^p \to B^q$ is taut over $F_1 \pi_1: \mathcal{A}_o \times \mathcal{A}_1 \to B$. Conversely, it is easily seen that if $\Phi_1 \Pi_1$ is taut over $F_1 \pi_1$ then Φ_1 is taut over F_1. Thus Φ_1 is taut over F_1 iff $\Phi_1 \Pi_1$ is taut over $F_1 \pi_1$. The comma category (Φ_o, Φ_1) is the subequalizing category of $\Phi_o \Pi_o$ and $\Phi_1 \Pi_1$ which by theorem A above, is a top category over the subequalizing category of $F_o \pi_o$ and $F_1 \pi_1$, i.e.,

the comma category (F_0,F_1), if $\Phi_1\Pi_1$ is taut over $F_1\pi_1$, i.e., if Φ_1 is taut over F_1. These remarks can be summarized in the following

<u>Theorem B</u> <u>If $\Phi_0\colon \mathcal{A}_0^{p_0} \to \mathcal{B}^q$ and $\Phi_1\colon \mathcal{A}_1^{p_1} \to \mathcal{B}^q$ are two functors such that Φ_0 lifts $F_0\colon \mathcal{A}_0 \to \mathcal{B}$ and Φ_1 is taut over $F_1\colon \mathcal{A}_1 \to \mathcal{B}$ then the comma category (Φ_0,Φ_1) is a top category over the comma category (F_0,F_1).</u>

If $F\colon \mathcal{A} \to \mathcal{B}$ is a functor and $B_0 \in \mathcal{B}$ is a fixed object then let (B,F) stand for the category whose objects are triples (B,f,A) where $f\colon B \to FA$ is in $\mathcal{B}$ and whose morphisms $\alpha : (B,f,A) \to (B,f',A')$ are given by morphisms $\alpha : A \to A'$ of $\mathcal{A}$ such that $(F\alpha)f = f'$. The initial object in this category, if it exists, has been called the <u>universal morphism for F with domain B</u> (WYLER [172]). If now $\Phi\colon \mathcal{A}^p \to \mathcal{B}^q$ lifts $F\colon \mathcal{A} \to \mathcal{B}$ and $(B,y) \in \mathcal{B}^q_0$ then let $r(B,f,A) = \{x \in pA \mid y \leq f^q\varphi_A x\}$. Proceeding exactly as above, it can be shown that if Φ is taut over F, then r is a topological theory on (B,F) and the top category $(B,F)^r$ is isomorphic to $((B,y),\Phi)$. Since in general for any top category $\mathcal{A}^p$, the initial object is (A,α_A) where A is the initial object of $\mathcal{A}$, the following theorem is obvious.

Theorem C (WYLER [172]) *Let $\varphi: \mathcal{A}^p \to \mathcal{B}^q$ be taut over $F: \mathcal{A} \to \mathcal{B}$. Then $h: (B,y) \to \Phi\ (C,z)$ is a universal morphism for Φ iff $h: B \to FC$ is a univetsal morphism for F and*

$$z = \inf\{ x \in pC | y \leqslant h^q \varphi_C x\}$$

3.5 Adjunction and local adjunction

Suppose $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ lifts $F: \mathcal{A} \to \mathcal{B}$. If F has a left adjoint G, does Φ have a left adjoint which lifts G? The answer to this question is known.

Theorem A (WYLER [172]) *If $F: \mathcal{A} \to \mathcal{B}$ has a left adjoint $G: \mathcal{B} \to \mathcal{A}$ and $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ is taut over F then Φ has a left adjoint $\Psi: \mathcal{B}^q \to \mathcal{A}^p$ which lifts G. Conversely, if Φ has a left adjoint Ψ then Φ is taut over F and $G = P\Psi\alpha = P\Psi\omega$ is left adjoint to F.*

A functor $F: \mathcal{A} \to \mathcal{B}$ is said to be *locally left adjunctable* (KAPUT [108]) if for every $f: B \to FA$ in $\mathcal{B}$ there exists an object $fB \in \mathcal{A}$ and morphisms $f_1: fB \to A$ in $\mathcal{A}$, $f_o: B \to FfB$ in $\mathcal{B}$ such that $(Ff_1)f_o = f$. Moreover, if $f = (Ff_1')f_o'$ with $f_1': A' \to A$ then there exists a unique $\theta: fB \to A'$ such that $(F\theta)f_o = f_o'$ and $f_1'\theta = f_1$. Local reflections, as example of local adjunctions have been considered by KAPUT [107]. We briefly mention them in the appendix. Let now morph $\mathcal{A}$ be the comma

category $(1_{\mathcal{A}}, 1_{\mathcal{A}})$. Then there is obviously a functor $\overline{F}$: morph $\mathcal{A} \to (1_{\mathcal{B}}, F)$ defined as follows. If $f: A \to B$ is an object of morph $\mathcal{A}$, $\overline{F} f = (FA, Ff, B)$. If (h,k): $(A,f,B) \to (A',f',B')$ in morph $\mathcal{A}$ is a morphism then $\overline{F}(h,k) = (Fh,k)$. The dual of the following theorem is proved in KAPUT [108].

<u>Theorem B</u>: F <u>is locally left adjunctable iff</u> $\overline{F}$ <u>has a left adjoint</u>

Proceeding exactly as in the preceding section it can be easily seen that if $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ is taut over F then $(1_{\mathcal{B}^q}, \Phi)$ is a top category over $(1_{\mathcal{B}}, F)$. If both F and Φ are identity functors, the assumption is trivial Thus

<u>Proposition C</u> morph $\mathcal{A}^p$ <u>is a top category over</u> morph $\mathcal{A}$

<u>Proposition D</u> <u>The full subcategory of morph</u> $\mathcal{A}^p$ <u>given by monomorphisms (epimorphisms) of</u> $\mathcal{A}^p$ <u>is a top category over the full subcategory of morph</u> $\mathcal{A}$ <u>given by monomorphisms (epimorphisms) of</u> $\mathcal{A}$

If now $\Phi: \mathcal{A}^p \to \mathcal{B}^q$ lifts $F: \mathcal{A} \to \mathcal{B}$ then $\overline{\Phi}$: morph $\mathcal{A}^p \to$ morph $\mathcal{B}^q$ clearly lifts $\overline{F}$: morph $\mathcal{A} \to$ morph $\mathcal{B}$. Moreover

<u>Lemma E</u> Φ <u>is taut over</u> F <u>iff</u> $\overline{\Phi}$ <u>is taut over</u> F.

Proof: The structure maps $\bar{\varphi}_f$ are given by $\bar{\varphi}_f(x,y) = (\varphi_A x, \varphi_B y)$ where $x \in pA$, $y \in pB$ and $f: A \to B$ is an object in morph $\mathcal{A}$. Looking at them, the assertion is easily seen to be true.

Summarizing, we can state

Theorem F If Φ is taut over F and F is locally left adjunctable then Φ is also locally left adjunctable. Conversely, if Φ is locally left adjunctable then F is locally left adjunctable and Φ is taut over F.

3.6 An example

In view of theorem 3.5A, WYLER raised the following question: is it possible that Φ lifts F and has a left adjoint Ψ but not a left adjoint which lifts a functor $G: \mathcal{B} \to \mathcal{A}$? We show by an example below that this indeed is possible.

Let TOPAIR be the category whose objects are pairs (A,X) where A is a topological space and X is a subspace of A. A morphism $f: (A,X) \to (B,Y)$ in TOPAIR is a continuous function $f: A \to B$ such that $fX \subset Y$. If now A is a topological space let pA be the lattice of all its subspaces. For a continuous function $f: A \to B$, define $f^p: pB \to pA$ by setting $f^p Y = f^{-1}[Y]$ for $Y \in pB$. Then TOPAIR is TOP^p. If Q: TOP $\to$ ENS.

and P: TOP/IR $\to$ TOP are the projection functors then P lifts Q. Also P has a left adjoint α_p : TOP $\to$ TOP/IR However, α_p does not lift the left adjoint G to Q (or any other functor G : ENS $\to$ TOP.) since if A is a topological space, $P\alpha_p A = A$ but GPA is A with the discrete topology.

This example has been included in WYLER [172].

CHAPTER - IV

REFLECTIONS AND COREFLECTIONS

In this chapter, we obtain some characterizations of reflective and coreflective subcategories of a top category $\mathcal{A}^P$. The arguments employ familiar techniques and are mostly intended to demonstrate that in general, assumptions on $\mathcal{A}$ alone suffice for an examination of reflectivity in $\mathcal{A}^P$. In section one we obtain the Freyd-Isbell characterization of a reflective subcategory of $\mathcal{A}^P$ depending on the same in $\mathcal{A}$. We also obtain an improvement of the known characterization of isoreflective subcatogories of $\mathcal{A}^P$ (WYLER [172]) using some techniques of HUSEK [87]. Section two proves that GLEASON's method [54] of proving the coreflectivity of locally connected spaces in TOP is valid for characterizing arbitrary isocoreflective subcategories of any top category $\mathcal{A}^P$. The formulation in this section owes much to HERRLICH and STRECKER [80].

4.1 Reflections

The following theorem is known (c f. KENNISON [114])

Theorem A Let $\mathcal{A}$ be a category with products. Let (B_o, B_1) be a right bicategory structure on $\mathcal{A}$ such that $\mathcal{A}$ is B_o-colocally small. Then $\mathscr{C} \subset \mathcal{A}$ is B_o-reflective

<u>iff $\mathscr{C}$ is closed under products and B_1-subobjects.</u>

A right bicategory structure is just a bicategory structure except that B_o is no longer required to be a subclass of epimorphisms.

We have shown earlier (2.3B) that if (B_o, B_1) is a bicategory structure on $\mathcal{A}$ then (B_o^p , B_1^p) where

$$B_o^p = \{a:(A,x) \to (A',x') \mid a \in B_o\}$$

$$B_1^p = \{ a: (A, a^p x') \to (A',x') \mid a \in B_1\}$$

is a bicategory structure on $\mathcal{A}^p$. It is easily seen that the proof is valid for right bicategory structures as well. Moreover, if $\mathcal{A}$ is B_o-colocally small, clearly $\mathcal{A}^p$ is B_o^p - colocally small. And if $\mathcal{A}$ has products then by 2.2L, $\mathcal{A}^p$ also has products. Therefore, theorem A above is applicable to $\mathcal{A}^p$ and we have

<u>Theorem B</u> <u>Let $\mathcal{A}$ be a category with products. Let (B_o, B_1) be a right bicategory structure on $\mathcal{A}$ such that $\mathcal{A}$ is B_o-colocally small. If $\mathcal{A}^p$ is a top category over $\mathcal{A}$ and if (B_o^p , B_1^p) stands for the right bicategory structure induced on $\mathcal{A}^p$ then a subcategory $\mathscr{C} \subset \mathcal{A}^p$ is B_o^p - reflective iff it is closed under products and B_1^p-subobjects.</u>

This theorem is like theorem 7.4 of WYLER [172] Roughly, that theorem says that if $\mathcal{A}$ has the epi-

extremal mono factorization property, is colocally small and has products then $\mathscr{C}$ is epireflective in $\mathscr{A}^P$ iff it is closed under products and extremal subobjects. Now epimorphisms and extremal monomorphisms do not form a bicategory in general since extremal monomorphisms need not be closed under composition. Hence WYLER's result is apparently stronger in this sense. However, theorem B above employs the finer machinery of right bicategory structures which dispenses with the requirement of B_0 being a class of epimorphisms. Also, the proof in KENNISON [114] does not need that B_1 be closed under composition.

It is not hard to see that the characterization of arbitrary reflective subcategories (given in the appendix and due to KENNISON [114])can be similarly lifted into top categories using the same techniques.

We now proceed to discuss isoreflective subcategories. These are reflective subcategories such that the reflection morphism is of the form $(x, 1_A, x')$. Such a subcategory is automatically full. The following theorem closely resembles theorem 1 of HUSEK [87].

<u>Theorem C</u> <u>Assume that $\mathscr{A}$ has products. Then a replete subcategory $\mathscr{C} \subset \mathscr{A}^P$ is isoreflective iff the following three conditions hold.</u>

1) <u>The inclusion functor $\mathscr{C} \subset \mathscr{A}^p$ preserves products.</u>

2) <u>To every object (A,x) of $\mathscr{A}^p$ there corresponds an object (A',x') in $\mathscr{C}$ such that there exists a monomorphism $m: (A,x) \to (A',x')$.</u>

3) <u>If $m: (A,x) \to (A',x')$ is any monomorphism in $\mathscr{A}^p$ with (A',x') in $\mathscr{C}$ then there exists an $x_o \varepsilon$ pA such that $x \leqslant x_o \leqslant m^p x'$ and $(A,x_o) \varepsilon \mathscr{C}$.</u>

<u>Proof</u>: If $\mathscr{C}$ is isorfelctive then 1), 2), 3) are obviously satisfied. To see that the converse is true, let $\mathscr{C}$ satisfy 1), 2) and 3) and let (A,x) be any object in $\mathscr{A}^p$. Then there exists, by 2) and 3) an $x_o \varepsilon$ pA such that $x \leqslant x_o$ and $(A,x_o) \varepsilon \mathscr{C}$. Index all such x_o's by a set I and take the product $\prod_{i \varepsilon I} (A,x_i)$ with projections $\pi_i: \Pi(A,x_i) \to (A,x_i)$. Then there exists a unique morphism $m: (A,x) \to \Pi (A,x_i)$ such that $(x, p_i m, x_i) = (x, 1_A, x_i)$ and m can be easily verified to be a monomorphism. Then by 1) $\Pi(A,x_i)$ is in $\mathscr{C}$ and hence by 3) ther exists one $\vec{x} \varepsilon$ pA such that $x \leqslant \vec{x}$ and $(A, \vec{x}) \varepsilon \mathscr{C}$ with $m: (A, \vec{x}) \to \Pi (A,x_i)$ in $\mathscr{A}^p$. We claim that $1_A: (A,x) \to (A, \vec{x})$ is the desired isoreflection of (A,x) in $\mathscr{C}$. If $\alpha: (A,x) \to (A',x')$ is a monomorphism with $(A',x') \varepsilon \mathscr{C}$ then there exists a $j \overset{o}{\varepsilon}$ I such that $x \leqslant x_j$ and $p_j m = 1_A: (A,x) \to (A_j,x_j)$. But then this

factors through $(A, \bar{x})$. If $\alpha: (A,x) \to (A',x')$ is *not* a monomorphism, the same reasoning can be applied to the monomorphism given by the dotted arrow in the following diagram

$$\begin{tikzcd}
 & & (A,x) \\
(A,x) \arrow[urr, "1_{(A,x)}"] \arrow[rr, dashed] \arrow[drr, "\alpha"'] & & (A,x)\times(A',x') \arrow[u, "\mathrm{proj}"'] \arrow[d, "\mathrm{proj}"] \\
 & & (A',x')
\end{tikzcd}$$

Thus $\mathscr{C}$ is isoreflective.

We shall say that a class of objects $\mathscr{C} \subset \mathscr{A}^p$ is *hereditary* iff whenever $m: A \to B$ is a monomorphism and (B,y) is an object in $\mathscr{C}$, $(A, m^p y) \underset{o}{\varepsilon} \mathscr{C}$. This definition of 'hereditary' is slightly weaker then that given by WYLER [172] in as much as it replaces extremal monomorphisms by arbitrary monomorphisms.

Lemma D *Conditions 2) and 3) in theorem C above are equivalent to the following two conditions;*

2') *All objects (A, ω_A) are in $\mathscr{C}$*

3') *$\mathscr{C}$ is hereditary*

Proof: Assume that 2) and 3) in theorem C are satisfied. Then since (A,ω_A) is in $\mathscr{A}^p$ there exists $x' \varepsilon\, pA$ such that

$(A, x') \in \mathscr{C}$ and $\omega_A \leqslant x'$, i.e. all (A, ω_A) are in $\mathscr{C}$. Also, if $m: A \to A'$ is a monomorphism in $\mathcal{A}^P$ for any $x' \in pA'$, $m: (A, m^P x') \to (A', x')$ is a monomorphism in $\mathcal{A}^P$ and by 3) $\mathscr{C}$ is hereditary. Conversely, if 2') and 3') are satisfied then for any $(A, x) \in \mathcal{A}^P_o$, the existence of $1_A: (A, x) \to (A, \omega_A)$ satisfies 2) and the existence of $1_A: (A, x) \to (A, m^P x')$ with $(A, m^P x') \in \mathscr{C}_o$ for a monomorphism $m: (A, x) \to (A', x')$ where $(A', x') \in \mathscr{C}_o$, satisfies 3).

Combining, we get

<u>Proposition E</u> <u>If $\mathcal{A}$ has products then a subcategory $\mathscr{C}$ of $\mathcal{A}^P$ is isoreflective iff $\mathscr{C}$ is productive, hereditary and replete and all objects (A, ω_A) are in $\mathscr{C}$.</u>

This theorem was originally proved by KENNISON [112] for $\mathcal{A}^P$ = TOP (see appendix). WYLER [172] has improved it to the form of proposition E above. His result differs from proposition E in the meaning assigned to 'hereditary'.

4.2 <u>Coreflections</u>

In this section, we obtain a characterization of isocoreflective subcategories of $\mathcal{A}^P$. As mentioned earlier, the techniques of this section go back to GLEASON [54].

Lemma A Let L be a complete lattice, let $x \to x'$ be an endomorphism of L in ORD which is decreasing and let F be the set of fixed points of this endomorphism. Then there exists a unique decreasing endomorphism of L which retracts L onto F.

Propostion B A subcategory $\mathscr{C}$ of $\mathcal{A}^P$ is isocoreflective iff for each object (A,x) of $\mathcal{A}^P$ there exists one object $(A,\bar{x})$ satisfying the following four conditions

1) $\bar{x} \leqslant x$

2) $(A, \bar{x}) \underset{o}{\varepsilon} \mathscr{C}$

3) $\bar{x}$ is the greatest element of pA satisfying 1) and 2)

4) If $\alpha: (A,x) \to (A',x')$ is in $\mathcal{A}^P$ then $\alpha: (A,\bar{x}) \to (A',\bar{x}')$ is also in $\mathcal{A}^P$

Proof Obvious

The promised characterization is:

Theorem C $\mathscr{C}$ is an isocoreflective subcategory of $\mathcal{A}^P$ iff for each $A \underset{o}{\varepsilon} \mathcal{A}$ there exists a function $\alpha: pA \to pA$ such that

1) α is decreasing

2) the fixed points of α are precisely the $\mathscr{C}$ objects on A i.e. structures such that A, with those

<u>structures gives rise to objects which are in</u> $\mathscr{C}$

3) <u>If</u> $f\colon (A,x) \to (B,y)$ <u>is a morphism in</u> $\mathcal{A}^p$,

then so is $f\colon (A, ax) \to (B,by)$.

<u>Proof</u> Assume that $\mathscr{C}$ is isocoreflective. Then proposition B above says that $a\colon pA \to pA$ defined by setting $ax = \bar{x}$ is a function satisfying 1), 2) and 3) above.

Conversely, let $a\colon pA \to pA$ satisfy 1), 2) and 3) above. Then if (A, x_1), (A,x_2) are in $\mathcal{A}^p$ such that $x_1 \leqslant x_2$ then $1_A\colon (A, x_1) \to (A,x_2)$ is in $\mathcal{A}^p$ so that by 3), $1_A\colon (A, ax_1) \to (A,ax_2)$ is in $\mathcal{A}^p$ i.e. $ax_1 \leqslant ax_2$ and $a \in$ ORD. Therefore lemma A above is applicable and there exists a unique decreasing endomorphism $x \to x^*$ of pA which retracts pA to the set F of those structures $x \in pA$ for which $(A,x) \in \mathcal{A}_o$. We claim that $(A, x^*) \to (A,x)$ is the desired isocoreflection.

Since $x \to x^*$ is decreasing, $1_A\colon (A,x^*) \to (A,x)$ is a morphism in $\mathcal{A}^p$. Next, if $(B,y) \in \mathscr{C}_o$ and $f\colon (B,y) \to (A,x)$ is in $\mathcal{A}^p$ then denote $f\colon (B,y) \to (A,f_py)$ by $\hat{f}$. Then $\hat{f}$ is in $\mathcal{A}^p$ and by 3), $\hat{f}\colon (B,by) \to (A,af_py)$ is also in $\mathcal{A}^p$. However, since y is a fixed point of b, $by = y$. Then $\hat{f}\colon (B,y) \to (A,af_py)$ is in $\mathcal{A}^p$. By 2.2C, $1_A\colon (A, f_py) \to (A,af_py)$ is in $\mathcal{A}^p$ i.e. $f_py \leqslant a f_py$. But a is decreasing so that $f_py = a f_py$

and $(A, f_p y) \in \mathscr{C}_o$. Since $x \to x^*$ is in ORD and since 2.2C says that $1_A: (A, f_p y) \to (A, x)$ is in $\mathcal{A}^p$, we have that $(f_p y)^* \leqslant x^*$. Thus $1_A: (A, f_p y) \to (A, x^*)$ is in ORD. The morphism $(y, 1_A \hat{f}, x^*)$ is the coreflection of (y, f, x).

Theorem C has been included in [172].

We close with an interesting observation due to WYLER [172]. If $\mathcal{A}^p$ and $\mathcal{A}^q$ are top categories over $\mathcal{A}$, a functor $T: \mathcal{A}^p \to \mathcal{A}^q$ is called a <u>top functor</u> over $\mathcal{A}$ if T is taut over $1_{\mathcal{A}}$. $\mathcal{A}^p$ is called a <u>top subcategory</u> of $\mathcal{A}^q$ if $\mathcal{A}^p$ is a subcategory of $\mathcal{A}^q$ and the embedding functor is a top functor over $\mathcal{A}$. <u>Cotop functors</u> and <u>cotop subcategories</u> are dually defined. Then top (cotop) subcategories are precisely the isoreflective (isocoreflective) subcategories. Details are available in [172].

CHAPTER – V

FREE AND PROJECTIVE OBJECTS

A free abelian group may be defined as a coproduct of copies of the infinite cyclic group Z or else as an abelian group satisfying a certain universal property. The latter approach has found an expression in the language of adjoint functors and free-object functors as adjoints to the 'underlying set' functors are now well known. On the other hand, free objects are frequently seen to be the coproducts of a certain fixed object, e.g. free topological spaces (discrete spaces) are disjoint topological sums of the single point space. The purpose of this chapter is to emphasize that this is no accident. In section one, we give a 'coproduct-definition' of free objects and observe that this agrees with the usual 'adjoint-definition' in a fairly wide class of categories. In section two, projective objects have been defined for the purpose of having a theory in which free objects will be projective. Some familiar results about free and projective objects have been established. In section three, we give a characterization of the category of sets. This section is independent of the other two. Connecting all this to top categories is trivial; if $\mathcal{A}$ is concrete with the underlying set

functor $u: \mathcal{A} \to$ ENS then $\mathcal{A}^P$ is concrete with $Pu: \mathcal{A}^P \to$ ENS.

It is a conveneient place to point out that a 'coproduct-definition' of free objects has been given by SEMADENI [160]. As for as the author knows however, no attempt was made to connect it to the 'adjoint-definition'.

5.1 Free Objects

For a functor $T\ \mathscr{C} \to$ ENS, the category of elements of T, denoted $\underline{El}\,T$, is the category whose objects are pairs (x, A) where A is an object of $\mathscr{C}$ and $x \in TA$. A morphism $f: (x, A) \to (y, B)$ in $\underline{El}\,T$ is given by a morphism $f: A \to B$ of $\mathscr{C}$ such that $Tf(x) = y$. The initial object of $\underline{El}\,T$, if it exists, has been called the universal element for T (cf. MacLANE and BIRKHOFF [135] and we shall follow this termonology although "initial element" would now seen to be the most natural term. When $\mathscr{C}$ is concrete, i.e. there exists a faithful 'underlying set' functor $u: \mathscr{C} \to$ ENS it is customary to define, the free object functor Fr: ENS $\to \mathscr{C}$ as the left adjoint to u. When it exists, the objects $Fr\,X$ are said to be free on the set X. In the category of groups (abelian groups, modules, topological spaces, compact Hausdorff spaces) with the usual

underlying set functor, free objects are the usual free groups (free abelian groups, free modules, discrete spaces, Stone-Cech compactifications of discrete spaces).

Now $Fr: ENS \to \mathscr{C}$ is left adjoint to $u: \mathscr{C} \to ENS$ iff the following holds: to any set X, there exists an object FrX and a function $\xi: X \to uFrX$ such that for any object A of $\mathscr{C}$ and any function $f: X \to uA$, there exists a unique morphism $h: FrX \to A$ such that $(uh)\xi = f$.

<u>Lemma A</u> <u>The free object functor exists only if the underlying set functor has a universal element.</u>

<u>Proof</u>: Let $u: \mathscr{C} \to ENS$ be the underlying set functor and assume that its left adjoint, the free object functor $Fr: ENS \to \mathscr{C}$ exists. Consider the object $Fr(*)$ where $*$ stands for the single-point set. When we recall that an element in any set is simply a function with $*$ as its domain, the condition given above for adjointness, on putting $\xi(*) = b$, reads: To any object A and to any point $f \in uA$ there is a unique morphism $h: Fr(*) \to A$ such that $uh(b) = f$. In other words, $(b, Fr(*))$ is the universal element for u.

If (b,B) is the universal element for the underlying set functor $u: \mathscr{C} \to ENS$, let B be called the

universal free object. A free object in $\mathscr{C}$ is defined to be a coproduct of copies of B.

Proposition B Assume that $\mathscr{C}$ *has coproduct and* u *has a universal element* (b,B). *Then the free object functor exists. Conversely if the free object functor* Fr: ENS $\to \mathscr{C}$ *exists, then* u *has a universal element* (b,B) *and* FrX *is precisely the coproduct of* card X *copies of* B.

Proof: Define $\mathrm{Fr}X = \underset{x \,\varepsilon\, X}{+} B_x$ where B_x is a copy of B, and let $\beta_x \colon B_x \to + B_x$ be the canonical injections. Define $\xi \colon X \to u + B_x$ by setting $\xi x = u(\beta_x)(b_x)$ where b_x is the copy of b in uB_x.

$$\begin{tikzcd}
X \arrow[r, "\xi"] \arrow[dr, "f"'] & u(+B_x) \arrow[d, dashed, "uh"] \\
& uA
\end{tikzcd}
\qquad
\begin{tikzcd}
+B_x \arrow[d, dashed] & B_x \arrow[l] \arrow[dl, "f_x"] \\
A &
\end{tikzcd}$$

Next, let A be any object in $\mathscr{C}$ and let $f \colon X \to uA$ be any function. Then to each $x \,\varepsilon\, X$ there is a unique morphism $f_x \colon B_x \to A$ with $uf_x(b_x) = fx$. Consequently, there is a unique morphism $h \colon + B_x \to A$ with $h\beta_x = f_x$. Then $(uh)\ (\xi(x)) = (uh)\,(u\beta_x)\,(b_x) = u\,(h\beta_x)\,(b_x) =$

$u(f_x)(b_x) = fx$ for every $x \in X$, i.e. $uh\ \xi = f$. Thus Fr is left adjoint to u. The converse follows from lemma A and the facts that Fr, being left adjoint to u, preserves coproducts and a set X is the coproduct of card X copies of $*$.

A speculative digression may not be out of place here. Sometimes it is obvious that the underlying set functor u does not have a universal element but it is still possible to find a pre-initial set in $\underline{El}\,u$ i.e. a set (x_i, A_i), $i \in I$, of objects of $\underline{El}\,u$, such that to any object (y, B) of $\underline{El}\,u$ there exists some $j \in I$ for which there is a morphism $f: (x_j, A_j) \to (y, B)$ in $\underline{El}\,u$. Such functors are called <u>petty</u> (cf. FREYD [49]). The underlying set functor of the category of all torsion groups is petty with the set of all finite cyclic groups with their generators forming the pre-initial set. Direct sums of finite cyclic groups may be then called "pre-free" and a suitable pre-universal property of these objects is not hard to show.

5.2 Projective Objects

A morphism e of a concrete category $\mathscr{C}$ may be called a surmorphism if $u\,e$ is a surjection, where u is the underlying set functor. If e is a surmorphism

and $fe = ge$ then $u(fe) = u(ge)$ i.e. $uf \ ue = ug \ ue$ which implies that $uf = ug$. Since u is faithful, $f = g$ and e is indeed an epimorphism. With this definition, we say that an object X is <u>projective</u> if for any surmorphism $e : A \rightarrow B$ and any morphism $X \rightarrow B$, the dotted arrow in the following diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \arrow[dl, dashed] \\ A \arrow[r, "e"'] & B \end{tikzcd}$$

may be filled so as to render it commutative.

This definition is slightly different from the usual definition of projective objects in as much as it replaces the class of all epimorphisms with the class of surmorphisms. That this is necessary can be seen from the example of the category of Hausdorff spaces where epimorphisms are maps with dense images. The usual definition of projectivity then fails to satisfy our expectations; the most obvious free object the single point space-is not projective. Surjectivity then, is needed if we want free objects in familiar categories to be projective. Quite justifiably, it may be said that some thing more is needed if one

wants to get 'good' projective objects - this definition in Hausdorff spaces yields only discrete spaces as projectives. However, here projective objects are defined for the sole purpose of exhibiting a satisfactory relationship with the free objects defined above.

<u>Proposition A</u> <u>The universal free object is projective.</u>

<u>Proof</u>: Let $e: X \to Y$ be a surmorphism , and let $f: B \to Y$ be any morphism. It is easily verified that the covariant homfunctor $\mathscr{C}(B,-)$ is naturally equivalent to the underlying set functor $u: \mathscr{C} \to \mathrm{ENS}$. In particular, the following diagram

$$\begin{array}{ccc} uX & \cong & \mathscr{C}(B,X) \\ \downarrow{\scriptstyle ue} & & \downarrow{\scriptstyle \mathscr{C}(B,e)} \\ uY & \cong & \mathscr{C}(B,Y) \end{array}$$

commutes. Corresponding to $f \in \mathscr{C}(B,Y)$, pick the unique element $uf(b)$ in uY, then the existence of an $x \in uX$ with $ue(x) = uf(b)$ is guaranteed by the surjectivity of ue. Pick $h: B \to X$ corresponding to $x \in uX$ via the top natural bijection in the diagram above. Then evidently $f = eh$ and B' is projective.

Proposition B Projective objects are closed under coproducts

Proof: Let $e : Y \to Z$ be a surmorphism. If each X_i is projective and $x_i : X_i \to X$ is their coproduct, we wish to show that X is also projective. Let $f: X \to Z$ be any morphism. Then $fx_i : X \to Z$ are morphisms and since each X_i is projective there are morphisms $h_i : X_i \to Z$ such that $eh_i = fx_i$. Since X is the coproductr, there exists a unique morphism $h: X \to Y$ such that $hx_i = h_i$. Then $ehx_i = eh_i = fx_i$ and since x_i are canonical injections, $eh = f$. That means X is projective.

Corollary C A free object is projective

Proof: A free object is a coproduct of copies of the universal free object.

Proposition D For any object A, there exists a free object $\bar{A}$ and a surmorphism $e: \bar{A} \to A$.

Proof: Set $\bar{A} = FruA$ and let $\alpha: u\bar{A} \to uA$ be the injection. Then there exists $e: \bar{A} \to A$ such that $(ue)\,\alpha = 1_{uA}$. It follows that ue is surjective and e is a surmorphism.

Proposition E A retract of a projective object is projective.

Proof: Let $q: X \to X'$ be a retraction i.e. there is $p: X' \to X$ such that $qp = 1_{X'}$. We shall show that if X is projective, so is X'. Let $e: Y \to Z$ be a surmorphism and let $h: X' \to Z$ be any morphism. Then $hq: X \to Z$ is a morphism and since X is projective there exists $g: X \to Y$ such that $eg = hq$. Then $egp = hqp = h$ so that $gp: X' \to Y$ fills the bill and X' is projective.

Proposition F The following are equivalent

1) X is projective

2) If $e: A \to X$ is a surmorphism then X is a retract of A.

3) X is a retract of a free object

Proof: That 2) follows from 1) is clear. If 2) holds Proposition D tells us that there exists a free object $\bar{X}$ with a surmorphism $e: \bar{X} \to X$: this means that X is a retract of $\bar{X}$. Finally, if 3) holds, corollary C and proposition E imply 1).

5.3 The category of sets and functions

In proving proposition 1.B above we used the fact that every set is the coproduct of copies of the

single-point set $*$. Curiously enough, this property is almost sufficient to characterize ENS. Precisely, we have the following

Theorem <u>A category $\mathcal{A}$ is (naturally equivalent to a skeleton of) the category ENS of sets and functions iff the following conditions hold</u>

1) <u>$\mathcal{A}$ has coproducts,</u>

2) <u>There exists an object T such that every object of $\mathcal{A}$ is (isomorphic to) a coproduct of its copies and for every $X \in \mathcal{A}$, $\mathcal{A}(T,X)$ consists of nothing except these canonical injections.</u>

Proof: If $\mathcal{A} = $ ENS, 1), 2) are obvious. Conversely, if 1), 2) hold then if X is any object, there exists a set EX such that $t_x : T_x \to \coprod_{x \in EX} T_x = X$, where each T_x is a copy of T and the t_x denote canonical injections. Let this correspondence be the action of a functor $E: \mathcal{A} \to$ ENS, on objects. If $h: X \to Y$ is any morphism then define $Eh: EX \to EY$ by setting $Eh(x) = y$ iff $\alpha t_x 1_{yx} = t_y$ where $1_{yx}: T_y \to T_x$

$$\begin{tikzcd} X = \underset{x \in EX}{+} T_x \arrow[d] & & T_x \arrow[ll, "t_x"'] \\ Y = \underset{y \in EY}{+} T_y & & T_y \arrow[ll, "t_y"] \arrow[u, "1_{yx}"'] \end{tikzcd}$$

stands for the identity morphism from T_y to T_x. By assumption, each $ht_x 1_{yx}$ must be one of the t_y's. If now $Eh = Eh'$ then $Eh(x) = Eh'(x)$ for every $x \in EX$ i.e. $ht_x 1_{yx} = h't_x 1_{yx}$ i.e. $ht_x = h't_x$ for every $x \in EX$. Since the t_x are canonical injections, $h = h'$ and E is faithful.

To show that E is full, let $f: EX \to EY$ be any function. Let $T_x \to Y = T_x \to T_{fx} \to Y$. Then there exists a unique morphism $h: X \to Y$ such that

$$\begin{tikzcd} X \arrow[d, dashed, "h"] & T_x \arrow[l, "t_x"'] \arrow[dl, "q_x"] \\ Y & \end{tikzcd}$$

$ht_x = q_x$. Then by definition $Eh(x) = y$ iff $ht_x 1_{yx} = q_x 1_{yx} = t_y$ iff

$$T_y \xrightarrow{1_{yx}} T_x \xrightarrow{1_{xfx}} T_{f(x)} \xrightarrow{t_{fx}} Y = T_y \xrightarrow{t_y} Y$$

i.e. iff $f(x) = y$. This means that $Eh = f$ and E is full. Since for any set A, $E(\underset{\alpha \varepsilon A}{+} T_\alpha)$ is isomorphic to A, E is also representative.

Thus E is faithful, full and representative i.e. an equivalence and $\mathcal{A}$ is naturally equivalent to ENS.

APPENDIX

CATEGORICAL METHODS AND THE CATEGORY OF TOPOLOGICAL SPACES

As has been remarked in the introduction, category theory has deeply penetrated the realm of general topology. In the introduction we gave a general outline of how the problem of finding categories in which the concepts of 'topology on a set' and 'continuity of a function' have some worthwhile meaning has been attacked In this article, we shall point out how categorical methods have been used to reformulate, sharpen, clarify and improve our knowledge of the very familiar category of topological spaces. It will be necessary, by the very nature of the subject under discussion to talk of general hypotheses and general results but we shall attempt to point out how the theory applies to the particular category in question. No claim to completeness is made but it is hoped that the article gives an idea of the current situation of the topics treated- the "currentness" of course being open to question for obvious reasons. We must mention that a brief but remarkably broad survey has been given by HERRLICH [76].

Throughout the following pages a topological property is a full replete subcategory of TOP and

τ_i, $i = 0,1,$etc. stand for topological properties consisting of T_i-spaces.

1. Generalities

FREYD [47] showed that the automorphism class group of TOP is trivial. We shall not discuss the definition of automorphism class group, our sole concern is with the implication of this result which is that every thing about topological spaces can be known if one forgets one's toplogy but remember one's category theory. FREYD [47] has given a method by which one can recogonize 'open' and 'closed' sets. The following characterization of TOP is due to HUSEK [89].

A category $\mathscr{C}$ is equivalent to TOP iff there is an object D in $\mathscr{C}$ such that

1) $\mathscr{C}(D,D) = \{h,k, 1_D\}$ where $hk = h$, $kh = k$.

2) If $h_i \in \mathscr{C}(X,Y)$, $i = 1,2$, $h_1 \neq h_2$ then there is an $f \in \mathscr{C}(D,X)$ with the properties $fh = fk = f$, $h_1 f \neq h_2 f$. Let $< D,X >$ stand for the set $\{f: D \to X | fh = fk = f\}$

3) Assume that we are given functions $\psi: \mathscr{C}(X,D) \to \mathscr{C}(Y,D)$, $\varphi: < D,Y > \to < D,X >$ such that $f(\varphi g) = (\psi f)g$ for every g and f. Then $\varphi = \{f \to g \circ f\}$ for some $g: Y \to X$

4) Let $\varphi : \langle D, X \rangle \to \{h,k\}$ be a function which can be described by means of a family $\{g_j^i \mid i \in I,\ j \in j_i,$ j_i are finite$\}$ from $\mathscr{C}(X,D)$ in this way: $\varphi g = h$ iff $g_j^i\ g = h$ for some $i \in I$ and each $j \in J_i$. Then $\varphi = \{g \to f \circ g\}$ for some $f : X \to D$.

5) Let S be a set and Φ be a subset of ENS $(S, \{h,k\})$ satisfying the condition 4) with S and Φ instead of $\langle D, X \rangle$ and $\mathscr{C}(X,D)$. Then there is an X in $\mathscr{C}$ and a bijection $\varphi : S - \langle D, X \rangle$ such that the map $\{f \to \{x \to f\ \varphi\ x\}\} : \mathscr{C}(X,D) \to \Phi$ is also bijective.

Using a three-point space it is possible to give a similar characterization for closure and proximity spaces and presumally several other 'continuity structures' BENTLEY [20] has given necessary and sufficient conditions for a T-category (cf 1.1) to be isomorphis to TOP. His characterization depends on the fact that the lattice in question is atom- generated. SCHLOMIUK [158] has given an elementary theory, in the sense of LAWVERE [124], for TOP. It is to be noted that the auto-morphism class groups of τ_1 and τ_2 are trivial and similar characterization should be available. KOWALSKY [117] has given such a characterization for τ_1. Compact Hausdorff spaces, metric spaces, paracompact spaces etc. have been characterized by S.P. FRANKLIN and B.V.S. THOMAS (unpublished).

Description of monomorphisms and epimorphisms in subcategories of TOP is a matter of some interest. BURGESS [22] has given a general procedure by which one can find out epimorphisms in categories of topological spaces. Let X be a topological space and $A \subset X$ be a subset. Form the disjoint union $X^* = X_1+X_2$ where X_1 and X_2 are two copies of X and define an equivalence relation on X^* by identifying the two copies of A, let the quotient space be denoted by X_A. This construction can be used to prove the well known fact that epimorphisms in τ_2 are exactly those maps whose images are dense. Not surprisingly the characterization for epis which holds in τ_2 does not hold in all its subcategories; an example has been given by BURGGSS [22]. BARON [13]has used this construction to characterize epimorphisms in τ_0. His result is that a map $e: A \to B$ in τ_0 is an epimorphisms iff for each b in B, every neighbourhood of b intersects $\{b\}^- \cap eA$. The notion of'onto'in τ_2 is also a categorical notion. Thus,let an epimorphism f be called <u>pure</u> (ISBELL [96]) if $f = gh$, g monomorphism, implies that h is an epimorphism.Pure epimorphisms in τ_2 are just the continuous surjections.For monomorphisms, it is an open scandal that the usual definition of a monomporphism as a left cancellable morphism is not strong enough in TOP since subobjects do not turn out to be subspaces.Several attempts have been made to define 'good' monomorphisms. In 2.3 we mentioned the extremal monomorphisms of ISBELL [96]and the three identical concepts

of 'monomorphism' of ARDUINI [2], strict monomorphism of JURCHESCU and LASCU [103] and strong monomorphism of KELLEY [111]. In TOP they are all the same and give subspaces. In τ_2, they yield closed subspaces which is alright from the categorical point of view but the notion of an ordinary subspacs in τ_2 can also be categorically described. Explicitly, a monomorphism is <u>copure</u> (ISBELL [96]) if whenever $m = fe$ with e a pure epimorphism, e must be an isomorphism. Copure monomorphisms in τ_2 are precisely the topological embeddings.

A map $f: X \to Y$ in τ_2 is said to be <u>perfect</u> if it is a closed map and the inverse images of points in Y are compact. The notion of a perfect onto map is categorical. Although they don't explicitly say so, HENRIKSEN and ISBELL's [72] characterization of perfect onto maps amounts to the following A map $f: X \to Y$ is perfect iff the following diagram

i)
$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[d, "f"'] & \beta X \arrow[d, "\beta f"] \\ Y \arrow[r, "e_Y"'] & \beta Y \end{tikzcd}$$

where β is the Stone-Cech compactification functor and e_X, e_Y are the usual embeddings, is a pullback.

Several other useful concepts such as 'punktal' morphism appear in the systematic investigation by KOWALSKY [117].

2. Reflections and Coreflections

Before the categorical definition of reflections and coreflections appeared explicitly, the formal nature of most of the examples in general topology (as in other branches of mathematics) was wellknown. Thus CECH [25] and STONE [164] knew the universal properties of the Stone-Cech compactification which say that the full subcategory of compact Hausdorff spaces is epireflective in the category of completely regular Hausdorff spaces. The same is true of HEWITT's realcompact spaces [82], BANASCHEWSKI's zero-dimensional compact spaces [8] and a score of compactifications and completions that have appeared and are still appearing. Similarly, YOUNG [176] found out that every topological space has what turns out to be a coreflection in the category of locally connected spaces. GLEASON [54] rediscovered this fact and gave essentially explicit proofs that the full subcategory of locally connected spaces is indeed coreflective in TOP. ARHANGELSKII [3] did the same for k-spaces. FRANKLIN ([42], [43]) proved that his sequential spaces form a coreflective subcategory although the explicit statement was given by BARON [14]. The chain-net spaces (MISHRA [142]) are also coreflective. The monograph of HERRLICH

[73] gives a very lucid and complete account of all aspects of reflections and coreflections in topological spaces. Another detailed treatment is given by KANNAN [106]. For the sake of completeness and also to include some results which appeared later than [73] and [106], we sketch a brief summary of this theory in the remainder of this section.

It was FREYD who defined reflections. The reflection functor is of course just the left adjoint to the inclusion functor of the reflective subcategory and one may say that the definition is due to KAN [105] who defined adjoint functors. In any case FREYD [47] proved the following

<u>Theorem A</u> <u>Let $\mathscr{C}$ be a complete, locally and colocally small category and $\mathscr{A}$ a full subcategory replate in $\mathscr{C}$ such that $\mathscr{A}$ is closed under the formation of products and subobjects. Then $\mathscr{A}$ is reflective in $\mathscr{C}$.</u>

This has been improved by ISBELL, HERRLICH, BARON, KENNISON, EHRBAR and possibly others. The following result in KENNISON [114] is probably the current version of this theorem.

<u>Theorem B</u> <u>Let $\mathscr{C}$ be a category with products. Let (B_0, B_1) be a right bicategory structure on $\mathscr{C}$ such that $\mathscr{C}$ is B_0-colocally small. Then $\mathscr{A} \subset \mathscr{C}$ is B_0-reflective iff $\mathscr{A}$ is closed under product and B_1-subobjects.</u>

A right bicategory structure is just a bicategory structure except that B_o is not necessarily a subclass of epimorphisms.

This result, when applied to TOP with the bicategory structure given by B_o as continuous surjection and B_1 as topological embeddings (or to τ_2 with the bicategory structure given by taking B_o as maps with dense images and B_1 as closed embeddings) gives us all epireflective subcategories. Most of the wellknown examples in TOP are epireflective. This is not very surprising in view of the following

<u>Theorem C</u> <u>Every monoreflective subcategory of any category is also epireflective</u>

This can be found, for example, in HERRLICH [73] but is not hard to prove. KENNISON [112] classified reflective subcategories of TOP and τ_2 in three types. The subcategory is <u>simple</u> if each reflection morphism is one-one and onto. This is what we call isoreflective. The term simple has come to be used in another sense which we shall describe later. <u>Identifying</u> subcategories are epireflective subcategories of TOP whereas <u>embedding</u> subcategories are those which are epireflective in τ_2. KENNISON [112] asked whether these include all the reflective subcategories of TOP or τ_2. That they do

not, has been demonstrated by KENNISON [113] himself and HERRLICH [75], both of whom found out examples of reflective subcategories of τ_2 which are not epireflective. Both the constructions employ rigid spaces of de GROOT [61]. SKULA [63] gave another type. We examine his construction in some detail. Let (A,X) be a topological space. If $B \subset A$ is any subspace, let $bB = b_A B = b_{(A,X)} B$ denote the set of all points $x \in A$ with the following property: there exist no $G_1, G_2 \in X$ such that $p \in G_2 \sim G_1$ and $G_2 \cap B = G_1 \cap B$. Then b is a closure operator and defines a topology on A. A subspace $B \subset A$ is said to be b-dense iff $bB = A$. b-closure etc. are similarly defined. A reflective subcategory $\mathscr{C}$ of TOP is b-embedding iff each reflection morphism has a b-dense image. With this terminology SKULA [163] proves,

Theorem D A topological property $\mathscr{C}$ is b-embedding iff $\mathscr{C}$ is productive b-closed hereditary and $\mathscr{C} \subset \tau_0$.

If S is the two point space with three open sets and S^+ denotes the class of all b-closed subspaces of all products of copies of S, then S^+ is a reflective subcategory which is neither simple, nor identifying, nor embedding [163]. This example was also known to BARON [15] and HERRLICH [75].

It is pertinent to recall that BARON's characterization [13] of epimorphisms in τ_0 (see section one) is equivalent to saying that $e: A \to B$ is epimorphic in τ_0 iff eA is b-<u>dense</u> in B. By definition extremal monomorphisms in τ_0 are given by those continuous injection $m: A \to B$ for which whenever $A \xrightarrow{m} B = A \xrightarrow{e} Z \xrightarrow{f} B$ with eA b-dense in Z, A and Z are homeomorphic, that is, precisely by the inclusion maps of b-closed subspaces These epimorphisms and extremal monomorphism clearly define a bicategory structure on τ_0 and theorem D follows from theorem B.

We now have the following characterizations

<u>Theorem E (KENNISON [104]) A topological property is epi-reflective iff it is closed under products and subspaces.</u>

<u>Theorem F (KENNISON [104]) A Hausdorff topological property is epi-reflective iff it is closed under products and closed subspaces.</u>

<u>Theorem G (SKULA [163]) A T_0 topological property is epireflective iff it is closed under products and b-closed subspaces.</u>

All the three follow from the general theorem B above.

SKULA [163] also proved the following

Theorem H Let $\mathscr{C}$ be a reflective topological property. If $\mathscr{C} \not\subset \tau_0$ then $\mathscr{C}$ is isoreflective. If $\mathscr{C} \subset \tau_0$, $\mathscr{C} \not\subset \tau_1$ then $\mathscr{C}$ is b-embedding.

This result gives a classification of reflective topological properties upto τ_1-spaces. A complete classification of all reflective subcategories of TOP and its subcategories, is still, as for as I know, an open problem.

Isoreflective topological properties have also received considerable attention. To begin with we have KENNISON's characterization [112]

Theorem I A topological property $\mathscr{C}$ is isoreflective iff it is closed under product, subspaces and contains all the indiscrete spaces.

Another important characterization, for more general situations has been given by HUSEK [81]. His theorem 1 [81] when applied to TOP, reads as follows:

Theorem J A topological property $\mathscr{C}$ is iso-reflective iff the following four conditions hold

1) $\mathscr{C}$ is closed under products
2) Every topological space has a continuous injection into some space of $\mathscr{C}$.
3) For any topological space (A,x), the set of all toplogies on A which are in $\mathscr{C}$ and are larger than x, contains a down-cofinal subset.

4) Every continuous injection $m: (A,x) \to (B,y)$, where (B,y) is in $\mathscr{C}$, can be factorized as $(A,x) \to (A,x') \to (B,y)$ where (A,x') and $m: (A,x') \to (B,y)$ both are in $\mathscr{C}$.

Arbitrary reflective subcategories without any restrictions on the reflection morphisms have been characterized by KENNISON [114]. Call a morphism $f: X \to Y$ injective with respect to a subcategory $\mathscr{A}$, iff for all morphisms $g: X \to A$ with $A \in \mathscr{A}$, there exists at least one $h: Y \to A$ such that $hf = g$. The class of all morphisms of $\mathscr{C}$ which are injective with respect to $\mathscr{A}$ is denoted by $\Psi_{\mathscr{C}}\, \mathscr{A}$ or simply $\Psi_{\mathscr{A}}$. Call a bicategory structure (B_o, B_1) well-founded if the category $\mathscr{C}$ is B_o-colocally and B_1-locally small.

Then one has [114]

Lemma X Let (B_o', B_1') be a well-founded bicategory structure on the complete category $\mathscr{C}$. Let $\mathscr{A}$ be a full subcategory closed under the formation of products Let $\mathscr{B}$ be the full subcategory of all B_1'-subobjects of members of $\mathscr{A}$. Let $B_o = \Psi_{\mathscr{B}}\, \mathscr{A} \cap E_{\mathscr{B}}$ where $E_{\mathscr{B}}$ is the class of all epimorphisms of $\mathscr{B}$. Then

1) $B_o \subset B_1'$

2) If $\mathscr{B}$ is B_o-colocally small then there exists a unique B_1 such that (B_o, B_1) is a right bicategory structure on $\mathscr{B}$.

With the help of the preceding lemma, the following characterization of reflective subcategories can be obtained.

<u>Theorem L (KENNISON [114])</u> <u>Let $\mathscr{C}$ be a complete category and let (B_0', B_1') be a well-founded bicategory structure on $\mathscr{C}$. Let $\mathscr{A}$ be a replate full subcategory of $\mathscr{C}$. Then $\mathscr{A}$ is reflective in $\mathscr{C}$ iff the following these conditions hold</u>:

1) <u>$\mathscr{A}$ is closed under products</u>
2) <u>$\mathscr{B}$ is B_0-colocally small (where $\mathscr{B}$ and B_0 are defined as in the preceding lemma)</u>
3) <u>$\mathscr{A}$ is closed under the formation of $B_1' \cap B_1$ -subobjects the existence of B_1 guaranteed by the preceding lemma</u>.

Closely parallel results have been obtained by BARON [15]. For example,

<u>Theorem M</u> <u>Let $\mathscr{C}$ be a colocally small category with products in which every morphism can be factorized as an epi followed by a mono. Let $\mathscr{A}$ be a reflective subcategory of $\mathscr{C}$ and let $\mathscr{B}$ be the subcategory of $\mathscr{C}$ whose objects are $\mathscr{C}$-subobjects of objects in $\mathscr{A}$. Then $\mathscr{A}$ is a $\mathscr{B}$-epireflective subcategory of $\mathscr{B}$ and $\mathscr{B}$ is a $\mathscr{C}$ epiraflective subcategory of $\mathscr{C}$</u>

The subcategory $\mathscr{B}$ in theorem M above is an "intermediate subcategory" of $\mathscr{A}$ and $\mathscr{C}$. Under some conditions there exists greatest and smallest intermediate subcategories.

Theorem N (BARON [15]) Let $\mathcal{A}$ be a reflective subcategory of a locally small category $\mathcal{C}$ with intersections and equalizers. The subcategory $\mathcal{B}'$ whose objects are $\mathcal{C}$ extremal subobjects of the $\mathcal{A}$-objects is the smallest intermediate subcategory of $\mathcal{A}$ and $\mathcal{C}$.

If in addition, $\mathcal{C}$ is colocally small and complete, define $\mathcal{B}''$ to be the category such that $B'' \varepsilon \mathcal{B}''$ iff f_1, f_2: $RB' \to B''$, $f_1 rB' = f_2 rB'$ imply $f_1 = f_2$ where $B' \varepsilon \mathcal{B}'$, RB' is the reflection of B' in $\mathcal{A}$, and $rB' : B' \to RB'$ is the corresponding reflection morphism. Then $\mathcal{B}''$ is the largest intermediate category of $\mathcal{A}$ and $\mathcal{C}$.

Thus the smallest intermediate subcategory between compact Hausdorff spaces and TOP is the category of completely regular spaces.

The case of coreflections is, of course, dual to that of reflections. However, some results are available for them which warrant separate mention. In the first place, KENNISON [112] proved that for any coreflective topological property, the coreflection morphisms are bijective. This has been generalized to constant-generated categories by HERRLICH [77]. HERRLICH and STRECKER [80] have shown that the characterization of the coreflectivity of locally connected spaces given by GLEASON [54] is valid for arbitrary coreflective topological properties. They have also

[80] considered the relationship of covering etc. to coreflectivity. Finally, FRANKLIN [44] has given a theory of 'natural covers' which yields several coreflective categories, among them those of sequential and k-spaces, although it does not suffice for all the coreflective topological properties.

HERRLICH [74] has given a method which gives all the coreflective topological properties of TOP. Let l be an operator which associates to every pair (X,A) where $X \in \mathrm{TOP}_o$ and $A \subset X$, another subset $l_X A$ of X, called the set of l-limit points of A in X. l is called a __limit operator__ iff l satisfies the conditions 1), 2) and 3) below:

1) If $A \subset X$ then $A \subset l_X A \subset \bar{A}^X$.

2) If A and B are subsets of X then $l_X(A \cup B) = l_X A \cup l_X B$.

3) If $f: X \to Y$ is a map and $A \subset X$ then $f(l_X A) \subset l_Y(fA)$.

It is easily verified that any limit operator l satisfies 4) and 5) below.

4) If A and B are subsets of X then $A \subset B$ implies $l_X A \subset l_X B$.

5) If $f: X \to Y$ is a map and $A \subset Y$ with $l_Y A = A$ then $l_X(f^{-1}[A]) = f^{-1}[A]$.

An operator satisfying condition 5) is called a __pre-limit__ operator. A limit operator l is called

idempotent iff l satisfies condition 6) below

6) If $A \subset X$ then $l_X(l_X A) = l_X A$.

With any pre-limit operator there is associated a coreflective subcategory of TOP and conversely, with any coreflective subcategory of TOP there is associated an idempotent limit operator. This yields a 1.1 correspondence between idempotent limit operators and coreflective subcategories of TOP.

HERRLICH and STRECKER [79] give a nice summary of the theory of coreflections in a general context. For the topological case, KANNAN [106] is a very complete reference. KENNISON [115] has started applying coreflectivity results to the theory of covering spaces and fibrations.

Analogous investigations in locally convex spaces have been made by WILBER [170].

The problems of generation and intersectivity of reflective subcategories were raised, for the category of uniform spaces, by ISBELL [99]. If $\mathcal{A}$ is a class of objects let its epireflective hull be the smallest epireflective subcategory containing $\mathcal{A}$. This has also been called the span of $\mathcal{A}$ (cf. KENNISON [113]). The following characterization has been obtained by KENNISON, BARON, HERRLICH and possibly by several others.

<u>Theorem C</u> <u>Let $\mathscr{C}$ be a complete, locally and colocally small category and let $\mathcal{A}$ be a class of objects. The epireflective hull of $\mathcal{A}$ is precisely the full subcategory whose objects are extremal subobjects of products of objects of $\mathcal{A}$.</u>

PREUSS [154] gave a characterization of the epireflective hull when $\mathscr{C}$ is balanced. His construction depends on the notion of a <u>monomorphic family</u> which is a family $\{f_i : X \to X_i | i \in I\}$ with the property that if $h, k: Z \to X$ are two distinct morphisms then there exists a $j \in I$ such that $f_j h \neq f_j k$. Clearly a monomorphism constitutes a monomorphic family. The canonical morphisms of a limit (e.g. the canonical projections of a product) form a monomorphic family. A monomorphic family is precisely what TAYLOR [166] calls a family with left cancellation property. Some imbedding results of PREUSS [154] have also been given by TAYLOR [166] in a slightly different setting. If now $\mathcal{A}$ is a class of objects let $Q(\mathcal{A})$ be the full subcategory consisting of these objects X for which there is a monomorphic family, $f_i : X \to X_i$ with X_i in $\mathcal{A}$. If $\mathscr{C}$ is complete and locally as well as colocally small and $ER\mathcal{A}$ stands for the epireflective hull of $\mathcal{A}$, clearly $\mathcal{A} \subset ER\mathcal{A} \subset Q\mathcal{A}$. The main result of [154] is that if $\mathscr{C}$ is also balanced

then $E\,R\,\mathcal{A} = Q\mathcal{A}$. The result is used, interestingly enough, to point out that zero-dimensionality and total disconnectedness coincide in compact Hausdorff spaces <u>because</u> the category of compact Hausdorff spaces is balanced.

One of the most remarkable results in this area is the following theorem due to FRANKLIN [45].

<u>Theorem P</u> <u>Let $\mathcal{A}$ be a left-fitting full replete subcategory of $\tau_{3\frac{1}{2}}$ and let $\tau_2\mathcal{A}$ be its epireflective hull in τ_2. Then $X \underset{o}{\varepsilon} \tau_2\mathcal{A}$ iff there is a family of subsets of βX, each belonging to $\mathcal{A}$ whose intersection is X.</u>

This is a characterization in terms of the Stone-Cech compactification. The role played by β is explained by the fact that left fitting is defined in terms of perfect maps and perfect maps are characterized by the fact that (i) is a pullback. When β is replaced by an arbitrary epireflective functor $k\colon \tau_{3\frac{1}{2}} \to \kappa$, k-perfect maps and k-left fitting etc. can be defined. This has been done by FRANKLIN and HERRLICH [46] who besides giving the obvious analogoue of theorem P, obtain several other interesting results.

HERRLICH [75] calls a topological property <u>simple</u> if it is the epireflective hull of a single object. Thus compact Hausdorff spaces are simple in τ_2 and

$\tau_{3\frac{1}{2}}$ is simple in TOP. No categories between τ_1 and τ_2 can be simple in TOP and no categories between τ_2 and τ_3 can be simple in τ_2. Regular spaces are not simple in TOP. Some problems have been raised by HERRLICH [71]. One of them which asks whether his k-compact spaces are simple for every cardinal k, has been answered by HUSEK in affirmative who also generalized this problem and its answer to proximity spaces ([91], [92]) A long-standing conjecture in this area has been disproved recently by NYIKOS [147] who gave an example to show that a zero-dimensional real compact space is not necessarily N-compact. For a thorough discussion of the closely related concept of a 'universal object' we refer to BAAYEN [4].

For intersectivity one has the following

<u>Theorem Q</u> (BARON [15]) <u>Let $\{\mathcal{A}_i\}$ be a set of reflective subcategories of $\mathcal{C}$ where $\mathcal{C}$ is as in theorem O above. Then,</u>

1) <u>If each $\mathcal{A}_i$ is epireflective then $\cap \mathcal{A}_i$ is epireflective.</u>
2) <u>If $\mathcal{B}$ is colocally small and an intermediate category of $\mathcal{A}_i$ and $\mathcal{C}$ for each i, then $\cap\mathcal{A}_i$ is reflective.</u>
3) <u>If $\mathcal{B}_i$ is an intermediate category of $\mathcal{A}_i$ and $\mathcal{C}$ for each i and $\cap\mathcal{B}_i$ is colocally small then $\cap\mathcal{A}_i$ is reflective and $\cap\mathcal{B}_i$ is an intermediate category of $\cap\mathcal{A}_i$ and $\mathcal{C}$.</u>

The 'reflective hull' has been characterized by KENNISON [114].

Theorem R Let $\mathcal{C}$ be complete and let (B_0', B_1') be a well-founded bicategory structure on $\mathcal{C}$. Let $\mathcal{A} \subset \mathcal{C}$ be replete and productive. Let $\mathcal{B} \subset \mathcal{C}$ be the subcategory consisting of B_1'-subobjects of members of $\mathcal{A}$. Let $B_0 = \Psi_{\mathcal{B}}\mathcal{A} \cap E_{\mathcal{B}}$ where $\psi_{\mathcal{B}}$ and $E_{\mathcal{B}}$ are as in lemma K. Assume that $\mathcal{B}$ is B_0-colocally small and let (B_0, B_1) be the right bicategory structure (guaranteed by the same lemma) on $\mathcal{B}$. Then $\mathcal{A}^*$, the full subcategory of all B_1-subobjects of members of $\mathcal{A}$ is the smallest replete reflective subcategory for which $\mathcal{A} \subset \mathcal{A}^*$.

Proposition S (KENNISON [114]) With the notation in theorem R above, let $X \in \mathcal{B}_0$ be given. The following three statements ar equivalent:

1) $X \in_0 \mathcal{A}^*$
2) $f: X \to Y$ and $f \in B_0$ imply f is an equivalence.
3) $f: X \to Y$ and $Y \in_0 \mathcal{B}$ imply $f \in B_1$.

The 'almost' reflective character of some well known subcategories has been recently observed. Extremally disconnected spaces and H-closed spaces of ALEXANDROV-URYSOHN [1] are two examples. KATETOV [109] proved that every Hausdorff space X can be embedded in an H-closed space τX. Similarly,

ILIADIS [94] proved that for every Hausdorff space X there exists an extremally disconnected space σX and a surjection $\sigma_X: \sigma X \to X$ MIODUSZEWSKI and RUDOLF [141] have proved that τ, and in a modified form, σ are functors; functors from the category of Hausdorff spaces and <u>skeletal</u> maps (which are maps $f: X \to Y$ such that $\operatorname{int} f^{-1}(cl\,V) \subset cl f^{-1}(V$ for every open subset V of Y) to the full subcategories of H-closed and extremally disconnected spaces respectively; and what is more, τ is reflective functor and σ is a coreflective functor. In addition, they made the following observation.

1) The diagram

$$\begin{tikzcd}
\sigma X \arrow[r, "\tau_{\sigma X}"] \arrow[d, "\sigma_X"'] & \tau\sigma X = \sigma\tau X \arrow[d, "\sigma_{\tau X}"] \\
X \arrow[r, "\tau_X"] & \tau X
\end{tikzcd}$$

is both a pullback and a pushout)

2) the fact that it is a pullback enables us to reduce the construction of the Iliadis coreflection σX for an arbitrary Hausdorff space X to the particular case of H-closed spaces.

ILIADIS [94] proved that for every Hausdorff space X there exists an extremally disconnected space σX and a surjection $\sigma_X \colon \sigma X \to X$ MIODUSZEWSKI and RUDOLF [141] have proved that τ, and in a modified form, σ. are functors; functors from the category of Hausdorff spaces and <u>skeletal</u> maps (which are maps $f\colon X \to Y$ such that $\operatorname{int} f^{-1}(cl\, V) \subset cl f^{-1}(V$ for every open subset V of Y) to the full subcategories of H-closed and extremally disconnected spaces respectively; and what is more, τ is reflective functor and σ is a coreflective functor. In addition, they made the following observation.

1) The diagram

$$\begin{tikzcd}
\sigma X \arrow[r, "\tau_{\sigma X}"] \arrow[d, "\sigma_X"'] & \tau\sigma X = \sigma\tau X \arrow[d, "\sigma_{\tau X}"] \\
X \arrow[r, "\tau_X"] & \tau X
\end{tikzcd}$$

is both a pullback and a pushout)

2) the fact that it is a pullback enables us to reduce the construction of the Iliadis coreflection σX for an arbitrary Hausdorff space X to the particular case of H-closed spaces.

3) The fact that it is a pushout may be used in constructing the Katetov reflection τX for any Hausdorff space X if one knows how to do it for extremally disconnected spaces. (Unfortunately in a seminar held in July 1971 at the Indian Institute of Technology, Delhi, it was found out that at least the first half of [141] needs a good deal of gap-filling in proofs).

The epireflectivity of H-closed spaces, using different morphisms, has also been proved by VELIČKO [167] and HERRLICH and STRECKER [78].

Perhaps one of the most interesting results in this area is the construction of the Wallman compactification as a reflector by HARRIS [67]. Explicitly, a morphism $f: X \to Y$ of τ_1 is a wo-map if when ν is a finite open cover of Y, then there is a finite open cover μ of X such that for each $W \varepsilon \mu$ there is $V \varepsilon \nu$ with $cl\, fA \subset V$ whenever $A \subset X$ and $cl\, A \subset W$. Then the Wallman compactification is an epireflection from T_1-spaces and wo-maps to compact T_1-spaces and wo-maps.

'Local reflectivity' has also been investigated by several authors. Thus a full subcategory $\mathcal{B}$ of $\mathcal{C}$ is said to be <u>locally reflective</u> in $\mathcal{C}$ (KAPUT [107]) if whenever $f: A \to B$ is a morphism with $B \varepsilon \mathcal{B}$, where exists
o

an object $\bar{B} \in_{o} \mathcal{B}$ and morphisms $f_o: A \to \bar{B}$, $f_1: \bar{B} \to B$ such that $f = f_1 f_o$ and furthermore, whenever B' is another object in $\mathcal{B}$ with $f'_o: A \to B'$ and $f' \to B$ such that $f'_1 f'_o = f$ there exists a unique morphism $\bar{f}: \bar{B} \to B'$ such that $\bar{f} f_o = f'_o$. Locally reflective subcategories are closed under intersection but not necessarily so under arbitrary limits. Not much of the theory of reflections therefore, is likely to be extended automatically to local reflections. KAPUT [107] has announced a number of results which are mainly concerned with relating local reflections to reflections by means of suitable functors such as these with adjoints. The details, however, have not been available to the author.

BARON's quasi-epireflective subcategories [15] also have a local flavour. A subcategory $\mathcal{B}$ of $\mathcal{C}$ is <u>quasi-epireflective</u> in $\mathcal{C}$ if whenever $f: A \to B$ is a morphism with $B \in_{o} \mathcal{B}$ there is an object $B' \in_{o} \mathcal{B}$ and morphisms $f_o: A \to B'$ and $f_1: B' \to B$ such that $f_1 f_o = f$ and f_o is an epimorphism. If $\mathcal{C}$ is $\mathcal{B}$ colocally small and has products then $\mathcal{B}$ is epireflective in $\mathcal{C}$ iff $\mathcal{B}$ is productive and quasi-epireflective.

Lastly, we must mention <u>stable reflectors</u> of HARRIS [67]. Let $\mathcal{C}$ be a category and let $\mathcal{D}$ be a full subcategory. The morphism $r: C \to D$ is a $\mathcal{D}$-reflector

if $D \in \mathscr{D}$ and whenever $E \in \mathscr{D}$ and $s: C \to E$ then there is $t: D \to E$ with $ts = r$ (no uniqueness assumptions for t). The morphism $r: C \to D$ is stable if whenever $h: D \to D$ and $hr = h$ then $h = 1_D$. $\mathscr{D}$ is stably reflective if every $C \in \mathscr{C}_o$ has a stable reflector $r_C \in \mathscr{D}$. Stable reflectors are categorically unique. If ω-extension of a map between T_1-spaces is an extension of it to a map between the Wallman compactifications of its domain and codomain and a ω-map is a map which has a (not necessarily unique) ω-extension, then compact T_1-spaces and ω-maps are stably reflective, with the Wallman compactification as the stable reflector, in T_1-spaces and ω-maps.

3. Projectivity and injectivity

GLEASON [53] was probably the first to consider projective objects in a category other than those in algebra. (This of course has to be loosely interpreted- the category of compact Hausdorff spaces and continuous maps which he considered, is algebraic in the modern sense, e.g. see MANES [136]. For a precise treatment of this question, we refer to [81])He proved that projective compact Hausdorff spaces are precisely the extremally disconnected ones. A sharper proof of this fact was provided by RAINWATER [156]. Another proof using only

linear analysis, has been given by WRIGHT [171]. Extremally disconnected spaces again. GLEASON [53] proved, are the projective objects in the category of locally compact Hausdorff spaces and perfect maps. The assumption that the spaces are Hausdorff and projective objects are perfect-onto-projective comes natural to a meaningful theory and we shall consider these assumptions as basic unless otherwise stated.

With these hypotheses, FLACHSMEYER [41] proved that in more general categories e.g. those of regular and completely regular spaces, projective objects are once more just the extremally disconnected ones. For regular spaces his results have been confirmed by STRAUSS [165] which were in their turn confirmed by HASUMI [68] whose techniques are quite different. MIODUSZEWSKI and RUDOLF [140] proved that the results of FLACHSMEYER [41] are formally deducible from those of GLEASON [53] if one remembers the characterization of perfect onto maps given by HENRIKSEN-ISBELL [72] (cf. section one). Practically all these results were also obtained by ILIADIS [94] and PONOMAREV [150] whose approach is very lucidly explained in the survey by ILIADIS and FOMIN [95].

Another generalization, in a somewhat different sense, is due to COHEN [29]. Recall that a compact Hausdorff space is extremally disconnected iff it is the stone space of a complete Boolean algebra. Analogously, let it be k-extremally disconnected iff it is the stone space of a k-complete Boolean algebra where k is an infinite cardinal. A k-set, is a set whose cardinality does not exceed k. A k-open subset of a compact Hausdorff space X is the union of a k-family of cozero sets of X. A compact Hausdorff space in which every k-open subset has a k-open complement is called a k-space. These, of course, are very much different from the k-spaces of ARHANGELSKII [3] which quite properly contain them. A map $f: X \to Y$ is a k-map if given k-open U in X, there exists a k-open V in f(X) such that $f(cl\ u) = cl\ V$. A map $f: X \to Y$ is called irreducibleif it is a surjection but fA is properly contained in Y for each closed proper subset A of X. In the category of k-spaces and irreducible k-maps, projective objects are precisely the k-extremally disconnected ones.

A systematic investigation of the theory of projective objects has been given by BANASCHEWSKI [10] Below we give a brief outline of his work.

For a class $\mathcal{A}$ of morphisms of a category $\mathcal{C}$, he considers $\mathcal{A}$-projective objects. These are the usual projective objects with $\mathcal{A}$ replacing the class of epimorphisms. The conditions on $\mathcal{A}$ and $\mathcal{C}$ are the following.

1) $\mathcal{A}$ is closed under composition
2) If $f \in \mathcal{A}$ is a right inverse of $g \in \mathcal{A}$ then f is an isomorphism and conversely.
3) For any $f \in \mathcal{A}$ there exists a $g \in \mathcal{C}$ such that $f\,g\,h \in \mathcal{A}$ implies $h \in \mathcal{A}$ for all $h \in \mathcal{C}$ and $f\,g \in \mathcal{A}$.

An $f \in \mathcal{A}$ is called <u>essential</u> iff $f\,g \in \mathcal{A}$ implies $g \in \mathcal{A}$ for all $g \in \mathcal{C}$, and $\mathcal{A}^*$ is the class of all essential $f \in \mathcal{A}$. All isomorphisms of $\mathcal{C}$ are, by 2) in $\mathcal{A}$ and essential and 3) means that for each $f \in \mathcal{A}$ there exists a $g \in \mathcal{C}$ such that $f\,g \in \mathcal{A}^*$. If $f: X \to Y$ belongs to $\mathcal{A}^*$ and X is $\mathcal{A}$-projective then f (or sometimes X) is called an <u>$\mathcal{A}$-projective cover</u> of Y.

4) $\mathcal{C}$ has pullbacks and these preserve $\mathcal{A}$.
5) Any well-ordered inverse system in $\mathcal{A}$ has an upper bound in $\mathcal{A}$.

With these axioms, his main results are the following

Proposition A The following are equivalent:

1) X is $\mathcal{A}$ projective.

2) Any $f: Y \to X$ in $\mathcal{A}$ has a right inverse.

3) Any $f: Y \to X$ in $\mathcal{A}^*$ is an isomorphism.

Proposition B Any $X \in \mathcal{C}$ for which the class of all $E \to X$ in $\mathcal{A}^*$ is a set, has an $\mathcal{A}$ projective cover.

He also proves the expected uniqueness of $\mathcal{A}$-projective covers.

Let now $\mathcal{C}'$ be a subcategory of $\mathcal{C}$ such that $\mathcal{A} \subset \mathcal{C}'$. Further, let the following conditions be satisfied:

E1) If $f\,g \in \mathcal{C}'$ then $g \in \mathcal{C}'$ for any $f, g \in \mathcal{C}$

E2) $\mathcal{C}'$ is $\mathcal{A}^*$-left fitting i.e. $f: X \to Y$ in $\mathcal{A}^*$ and $Y \in \mathcal{C}'$ imply that $X \in \mathcal{C}'$.

Proposition C Projective objects in $\mathcal{C}'$ are the same as the $X \in \mathcal{C}'$ projective in $\mathcal{C}$, and for any $X \in \mathcal{C}'$, $X \in \mathcal{C}'$, an $\mathcal{A}$-projective cover $Y \to X$ is also one in $\mathcal{C}$ and conversely.

These results are then applied to the category of Hausdorff spaces. There the perfect onto maps satisfy 1) - 5) and if $f: X \to Y$ is perfect onto then card $X \leqslant 2^{2^{\text{card } Y}}$. With the assistance of a number

of lemmas and minor propositions, some of them due to others, he proves the following

Theorem D Let $\mathcal{B}$ be a subcategory of τ_2 which is closed with respect to pullbacks and such that for any $X \in \mathcal{B}$ and closed subsets A, B of X, $A + B$ and the natural morphism $A+B \to X$ belong to $\mathcal{B}$. Then the perfect-onto-projectives in $\mathcal{B}$ are exactly the extremally disconnected spaces belonging to $\mathcal{B}$.

Moreover, if ΛX stands for the space of convergent maximal open filters on X and $\Lambda^* X$ for the space obtained from this through enlarging the topology by adding the sets $\{M \mid \lim M \in U,\ M \in \Lambda X$, for U open in $X\}$ to the open sets of ΛX, then for each $X \in \mathcal{B}$ the limit mapping $\lim: \Lambda^* X \to X$ is a projective cover of X.

In addition, one has that $\Lambda^* X = \Lambda X$ iff X is regular (FLACHSMEYER [41])

Speaking of projective covers, PARK [148] has proved that if X is a completely regular Hausdorff space, the projective cover of its Stone Cech compactification βX is homeomorphic to the maximal ideal space, endowed with the Stone topology, of the maximal ring of quotients of the ring CX of all real valued continuous functions on X.

Some subcategories of τ_2 to which theorem D applies are:

1) Compact spaces,

2) locally compact spaces,

3) paracompact spaces,

4) realcompact spaces

5) σ-compact spaces

6) completely regular spaces

7) regular spaces

8) zero-dimensional spaces

and any intersection of these.

The relationship to reflections is exhibited in the folowing

Proposition E If an epireflective subcategory $\mathscr{C}$ of a subcategory $\mathscr{D}$ of τ_2 preserves perfect maps then $X \underset{o}{\varepsilon} \mathscr{D}$ is projective iff its reflection in $\mathscr{C}$ is projective in $\mathscr{C}$; moreover if $\mathscr{D}$ is also closed hereditary, then for any projective cover $f: X \to Y$ in $\mathscr{D}$ its reflection is a projective cover in $\mathscr{C}$.

With a slightly different class of maps, projectivity has been considered by MIODUSZEWSKI and RUDOLF [141]. Let a map $f: Y \to X$ be regular-open minimal if the topology in Y is generated by sets $f^{-1}[U] \cap V$ where U is open in X and V is regular open in Y. If $\mathcal{A}$ now stands for the class of regular-open minimal surjections then every $\mathcal{A}$-projective object

in the category of H-closed spaces and continuous functions is extremally disconnected although the converse is false. However, if an H-closed space (X,t) is <u>compact-like</u> in the sense that there exists a compact topology t' on X such that $t' \subset t$ and $cl_{t'}V' = cl_t V'$ for every $V' \in t'$, then it is $\mathcal{A}$-projective iff it is extremally disconnected. Several other interesting results about projectivity in H-closed spaces and Hausdorff spaces appear in [141].

The dual notion of injective objects has not been studied so intensively. MIODUSZEWSKI and RUDOLF [141] give some results. First, if $\mathcal{B}$ stands for the class of all open dense embeddings then in the category of extremally disconnected Hausdorff spaces and continuous maps every $\mathcal{B}$-injective object is H-closed. The converse is true when skeletal maps alone are taken as morphisms. For general Hausdorff spaces, the situation is as follows: Call a map $f: X \to Y$ in the category of Hausdorff spaces and skeletal maps, <u>σ-coperfect</u> iff the diagram

$$\begin{tikzcd}
\sigma X \arrow[r, "\sigma_X"] \arrow[d, "\sigma f"'] & X \arrow[d, "f"] \\
\sigma Y \arrow[r, "\sigma_Y"'] & Y
\end{tikzcd}$$

is a pushout where σ is the Iliadis coreflection functor and f is an open dense embedding. If a σ-injective object is an object injective with respect to σ-coperfect maps then a Hausdorff space is σ-injective iff it is H-closed.

In view of the Stone duality, projectivity in categories of lattices is not without interest for the present discussion. BALBES [7], BANASCHEWSKI-BRUNS ([11],[12]) HALMOS [66] are some of those who have discussed this subject. Also, though we are not concerned here with toplogical algebra or functional analysis, it is not out of place here to note that projectivity and injectivity have been intensively studied in these fields. BANASCHEWSKI ([9],[10]), COHEN [30], GRAEV [57], HALL [65], ISBELL ([97], [98], [100]), LACEY and COHEN [120], MARKOV [139], POTHOVEN [153], SEMADENI [160] and WARD [169] are some of the references where free, projective and injective objects in topological algebra and functional analysis have been studied. Concepts like 'exactness' etc. have also been formalized in functional analysis and one may visualize a fullscale program of homological algebra in topological vector spaces.

B I B L I O G R A P H Y

[1] ALEXANDROV, P. and URYSCHN, P. <u>Zur Theorie der Topologischen Raume</u>
Math. Ann. <u>92</u> (1924) 258-266

[2] ARDUINI, P. <u>Monomorphisms and epimorphism in abstract categories</u>
Rendi. Sem. Mat. <u>42</u> (1969) 135-166

[3] ARHANGELSKII, A. <u>Bicompact sets and the topology of spaces</u>
Soviet Math. Dokl. <u>4</u> (1963) 561-564

[4] BAAYEN, P.C. Universal Morphisms.
Math. Centre Tracts 9.
Amsterdam 1964

[5] BACSICH, P.D. <u>Compact injections and non-standard analysis</u>
Notices AMS <u>17</u> (1970) 836

[6] BAIANU, I. <u>Organismic supercategories II: on multi-stable systems</u>
Bull. Math. Biophysics <u>32</u> (1970) 539-561

[7] BALBES, R. <u>Projective and injective distributive lattices</u>
Notices AMS <u>13</u> (1966) 740.

[8] BANASCHEWSKI, B. Uber nulldimensionale Raume
Math. Nachr. 13 (1955) 129-140

[9] ______________ Essential extensions and injectivity for metric and Banach spaces.
Canad. Math. Bull. 12 (1969) 116-118

[10] ______________ Projective covers in categories of topological spaces and topological algebras
Proceedings of the Kanpur Conf. on Topology 1968
Academic Press 1971

[11] ______________ and BRUNS, G. Categorical characterizations of the MacNeille completion
Arch. der Math. 18 (1967) 369-377

[12] ______________ Injective hulls in the category of distributive lattices
J. fur Math. 232 (1968) 102-109

[13] BARON, S. Note on epi in To
Canad. Math. Bull. 11 (1968) 503-504

[14] ______________ The coreflective subcategory of sequential spaces
ibid. 603-64

[15] ______________ Reflectors as composition of epireflectors
Trans. AMS 136 (1969) 499-508

[16] BARR, M. and BECK, J. Acyclic models and triples
336-344 in Proc. Conf. Categorical Alg. La Jolla 1965
Apringer-Verlag 1966

[17] ______________ Homology and standard constructions
245-335 in Lecture Notes in Math. 80
Springer-Verlag 1969

[18] BECK, J. On H-spaces and infinite loop-spaces
139-153 in Lecture Notes in Math. 99
Springer-Verlag 1969

[19] BENABOU, J. Introduction to Bicategories
1-77 in Lecture Notes in Math. 47
Springer-Verlag 1967

[20] BENTLEY, H. T-categories and some representation theorems
Univ. of New Mexico Tech. Report 172

[21] BRUMMER, G.C.L. Initial quasiuniformities Indag. Math. 31 (1969) 403-409

[22] BURGESS, W. The meaning of mono and epi in some familier categories Canad. Math. Bull. 8 (1965) 759-769

[23] BURGHELCA, A. Syntactic types and theory of categories Rev. Roum. Math. Pures et Appl. 12 (1967) 363-371

[24] CARTAN, H. and EILENBERG, S. Homological Algebra. Princeton 1956

[25] CECH, E. On bicompact spaces Ann. Math. 38 (1937) 823-844

[26] ________ Topological Spaces, Praha 1966

[27] CHANG, C.L. Fuzzy topological spaces J. Math. Analysis and Appl. 24 (1968) 182-190

[28] CHASE, S. and SWEEDLER, M. Hopf Algebras and Galois Theory Lecture Notes in Math. 97. Springer-Verlag 1969

[29] COHEN, H.B. The k-extremally disconnected Spaces as projectives Canad. J. Maths. 16 (1964) 253-260.

[30] ______________ *Injective envelopes of Banach spaces*
Bull. AMS 70 (1964) 723-726

[31] CSASZAR, A. *Foundations of general topology*
MacMillen 1963

[32] DAVIS, R. *Equational systems of functors*
92-109 in Lecture Notes in Math. 47 Springer-Verlag 1967

[33] DEDECKER, P. *Three dimensional nonabelian cohomology for groups*
32-64 in Lecture Notes in Math.92
Springer-Verlag 1969

[34] DIEUDONNE, J. *Recent developments in mathematics*
Amer. Math. 71 (1964)239-248

[35] EHRESMANN, C. Categories et Structures
Paris 1965

[36] EILENBERG, S. and KELLEY, G.M. *Closed Categories*
421-562 in Proc. Conf. Cat. Alg. La Jolla 1965
Springer-Verlag 1966

[37] EILENBERG, S. and MacLANE, S. *General theory of natural equivalences*
Trans. AMS 58 (1945)231-294

[38] EILENBERG, S. and WRIGHT, J. _Automata in general algebras_ Inf. and Control _11_ (1967) 452-470

[39] ________________ Algebraic categories and automata theory To appear

[40] FEFERMAN, S. _Set theoretical foundations of category theory_ 201-232 in Lecture Notes in Math. 106 Springer-Verlag 1969

[41] FLACHSMEYER, J. _Topologischa Projecktivraume_ Math. Nachr. _26_ (1963)57-66

[42] FRANKLIN, S.P. _Spaces in which sequences suffice_ I Fund. Math. _57_ (1965) 107-115

[43] ________________ Spaces in which sequences suffice II ibid _61_ (1967) 51-56

[44] ________________ _Natural covers_ Composito Math. _21_ (1969) 253-261

[45] ________________ _On epireflective Hulls_ Gen. Top. and Appl. _1_ (1971) 29-31

[46] ______________ and HERRLICH, H. — *On epireflective Hulls II* to appear.

[47] FREYD, P. — Abelian categories Harper and Row 1964

[48] ______________ — *Stable Homotopy* 121-172 in Proc. Conf. Cat. Alg. La Jolla 1965 Springer-Verlag 1966

[49] ______________ — *Several new concepts* 196-241 in Lecture Notes in Math. 99 Springer-Verlag 1969

[50] GABRIEL, P. and ZISMAN, M. — Calculus of fractions and homotopy theory Springer-Verlag 1967

[51] GIVE'ON, Y. — *Transparent categories and categories of transition systems* 317-330 in Proc. Conf. Cat. Alg. La Jolla 1965 Springer-Verlag 1966

[52] ______________, and ARBIB — *Algebra Automata II* Inf. and Control 12 (1968) 346-370

[53] GLEASON, A.M. Projective topological spaces
Illinois J. Math. 2 (1958) 482-489

[54] ____________ Universal locally connected refinements
ibid. 7 (1963) 521-531

[55] GOGUEN, J.A. L-fuzzy sets
J. Math. Analysis and Appl. 18 (1967) 145-174

[56] ____________ Categories of V-sets.
Bull. AMS 75 (1969) 622-624

[57] GRAEV, M.I. Free topological groups
AMS Translations ser-one-8 (1962) 305-364

[58] GRAVES, W.. A categorical approach to combinatorial geometry
J. combinatorial theory.
To appear.

[59] GRAY, J. Fibered and cofibered categories
21-83 in Proc. Cof. Cat. Alg.
La Jolla 1965
Springer-Verlag 1966

[60] ____________ Some remarks on a categorical foundation of mathematics
306-312 in Lecture Notes in Math. 99
Springer-Verlag 1969

[61] de GROOT, J. *Groups represented by homeomorphism groups I* Math. Ann. 138 (1959) 80-102

[62] GROTHENDIECK, A. *Categories et descent* SGA, IHES, Paris 1961

[63] ________, with DIEUDONNE, J. Elements de geometic algebrique 1-V IHES, Paris

[64] HAJEK, O. *Categorical concepts in dynamical system theory* 243-257 in Topological Dynamics Benjamin 1968

[65] Hall, C.E. *Projective topological groups* Proc. AMS 18 (1967) 425-431

[66] HALMOS, P.R. *Injective and projective Boolean Algebras* Proc. Symp. Pure Math. 2 (1961) 114-122

[67] HARRIS, D. *Wallman compactification as a functor* Notices AMS 18 (1971) 208

[68] HASUMI, M. *A continuous selection theorem for extremally disconnected spaces* Math. Ann. 179 (1969) 83-89

[69] HEDRLIN, Z. PULTR, A. and TRNKOVA, V. — Concerning a categorical approach to topological and algebraic theories
176-181 in Gen. Top and relations to modern analysis and algebraII
Praha 1967

[70] HELLER, A. — Probabilistic automata and stochastic transformations
Math. Systems theory 1 (1967) 197-208

[71] ______ — Stable homotopy categories
Bull AMS 74 (1968) 28-63

[72] HENRIKSEN, M. and ISBELL, J. — Some properties of compactifications
Duke Math. J. 25 (1958) 83-106

[73] HERRLICH, H. — Topologische Reflexionen und coreflexionen
Lecture Notes in Math. 78
Springer Verlag 1968

[74] ______ — Limit operators and topological coreflections
Trans. AMS 146 (1969) 203-210

[75] ______ — On the concept of reflections in general topology
105-114 in Proc. Symp. Ext. Theory of Top. Struct. Berlin 1967.
Academic Press 1969

[76] ______________ Categorical Topology
Gen. Top. and Appl. 1 (1971) 1-15

[77] ______________ On constant maps in categories
To appear.

[78] ______________ and STRECKER, G. H-closed spaces and reflective subcategories
Math. Ann. 177 (1968) 302-309

[79] ______________ Coreflective subcategories I
Trans. AMS. To appear

[80] ______________ Coreflective Subcategories II
Rozprawy Mat. To appear

[81] ______________ Algebra ∩ topology = compactness
Gen. Top and Appl. To appear

[82] HEWITT, E. J. Rings of real valued continuous functions
Trans. AMS 64 (1948) 54-99

[83] HUBER, P. J. Homotopy Theory in general categories
Math. Ann. 144 (1961) 361-385

[84] HUSEK, M. S-categories
Comm. Math. Univ. Cardinae 5 (1964) 37-46

[85] ______________ Generalized proximity and uniform spaces I.
ibid. 247-266.

[86] ____________ Generalized proximity and unfiform spaces II
Comm. Math. Univ. Carolimae
6 (1965) 119-139.

[87] ____________ Remarks on Reflections
ibid 7 (1966) 249-259

[88] ____________ One more remark on reflections
ibid 8 (1967) 129-137

[89] ____________ Construction of special functors and its applications
ibid 555-566

[90] ____________ Categorical methods in topology
190-194 in Proc. Sec. Symp.
Gen. Top Praha 1966, Praha 1967

[91] ____________ The class of k-compact-spaces is simple
Math. Z. 110 (1969) 123-126

[92] ____________ Generalizations of kaplanski's and Shrirota's theorems on k-compact spaces
ibid 111 (1969) 214-220

[93] ____________ Categorical connections between generalized proximity spaces and compactifications
127-132 in Proc. Symp. Ext. Theory of Top struct Berlin 1967
Acdemic Press 1969

[94] ILIADIS, S. — The absolutes of Hausdorff spaces
Soviet Math. Dokl 4 (1963) 295-298

[95] ______ and FOMIN, S. — The method of centered systems in the theory of topological spaces
Russian Math. Surveys 21 (1966) No.4, 37-62

[96] ISBELL, J.R. — Subobjects, adequacy, completeness and categories of algebras
Roz prawy Mat. 36 (1964)

[97] ______ — Six theorems about injective metric spaces
Comm. Math. Helv. 39 (1964) 65-76

[98] ______ — Injective envelopes of Banach spaces are rigidly attacked
Bull. AMS 70 (1964) 727-729

[99] ______ — Uniform spaces
Math. Surveys No.12 AMS 1964

[100] ______ — Three remarks on injective invelopes of Banach spaces
J. Math. Analysis Appl. 27 (1969) 516-518

[101] ______ — TOP and its adjoint relatives
Proc. Kanpur Conf. Topology, Kanpur 1968. Academic Press 1971

[102] JOYAL, A. Boolean algebras as functors
Notices AMS 16 (1969) 949

[103] JURCHESCU, M. and LASCU, A. Strict morphisms, Cantorian categories, completion functors (Romanian)
Stud. cerc. Math. 18 (1966) 219-234

[104] KALMAN, FALB and ARBIB, M. Topics in mathematical systems theory.
McGraw Hill 1969

[105] KAN, D. Adjoint functors
Trans. AMS 87 (1958) 294-328

[106] KANNA N.V. Coreflexive subcategories in Topology,
Ph.D. thesis, Madurai University 1970

[107] KAPUT, J.J. Locally reflexive subcategories
To appear.

[108] ____________ Non-universal adjunctions
To appear.

[109] KATETOV, M. On H-closed extensions of topological spaces
Casopis matem. fys. 72 (1947) 17-32

[110] ______________ On continuity structures and spaces of mappings
Comm. Math Univ. Carolinae 6 (1965) 257-278

[111] KELLEY, G.M. Monomorphisms, epimorphisms and pullbacks
J. Australian Math. Soc. 1 (1968) 124-142

[112] KENNISON, J.F. Reflective functors in general topology and elsewhere
Trans AMS 18 (1965) 303-315

[113] ______________ A note on reflection maps
Illinois J. Math. 11 (1967) 404-409

[114] ______________ Full reflective subcategories and generalized coving spaces
ibid 12 (1968) 353-365

[115] ______________ Coreflection maps which resemble universal coverings
46-75 in Lecture notes in math. 86
Springer-Verlag 1969

[116] KIM, H. and RATTRAY, B.A. Topologies on objects of categories
Notices AMS 16 (1969) 949

[117] KOWALSKY, H. *Kategorien Topologischer Raume* Math. Z. 77 (1961) 249-272

[118] KREIZEL, G. *Category theory and foundations of mathematics* 233-245 in Lecture Notes in Math. 106 Springer-Verlag 1969

[119] KRISHNAN, V.S. *Categories with base functors* Proc. Kanpur conf. topology, Kanpur 1968 Academic Press 1971

[120] LACEY, H. and COHEN, J.B. *On injective envelopes of Banach Spaces* J. Functional Analysis 4 (1969) 11 - 30

[121] LAMBEK, J. *Deductive systems and categories I* Math. systems theory 2 (1968) 287-318

[122] ______ Deductive systems and categories 76-122 in Lecture Notes in Math. 8 Springer-Verlag 1969

[123] ______ *Subequalizers* Canad. Math. Bull. 13 (1970) 337-349

[124] LAWVERE, F.W. *Functorial semantics of algebraic theories* Proc. Nat. Acad. Sc. 50 (1963) 869-872

[125] ______________ *The category of categories as a foundation for mathematics*
1-20 in Proc. Conf. Cat Alg.
La Jolla 1965
Springer-Verlag 1966

[126] ______________ *Categorical dynamics*
Notices AMS 14 (1967) 937

[127] ______________ Category-valued Higher order Logic
To appear

[128] LINTON, F.E.J. The functorial foundations of measure theory
Thesis, Columbia 1963

[129] ______________ *Some aspects of equational categories*
84-94 in Proc. Conf. Cat. Alg.
La Jolla 1965
Springer Verlag 1966

[130] LORRAIN, F. *Notes on topological spaces with minimum neighborhoods*
Amer. Math. Month 76 (1969)
616-627

[131] MacLANE, S. *Locally small categories and foundations of set theory*
25-43 in infinitistic methods
Oxford 1961

[132] ________________ Possible programs for categorists
123-131 in Lecture Notes in
Math. 86
Springer Verlag 1969

[133] ________________ Foundations for categories and sets
146-164 in Lecture Notes in
Math. 92
Springer verlag 1969

[134] ________________ One universe as a foundation
for category theory
192-200 in Lecture notes in
Math.106
Springer Verlag 1969

[135] ________________ and BIRKHOFF, G. Algebra
MacMillan 1967

[136] MANES, E. A triple-theoretic construction
of compact algebras
91-118 in Lecture Notes in
Math. 80
Springer Verlag 1969

[137] ________________ Minimal Subalgebras for dynamic
triples
419-447 in Lecture Notes in
Math.99
Springer-Verlag 1969

[138] MARINOS, P. *Fuzzy logic and its applications to switching systems* IEEE Trans. Comput. C18 (1969) 343-348

[139] MARKOV, A.A. *On free topological groups* AMS Translations ser one 8 (1962) 195-272

[140] MIODUSZEWSKI, J. and RUDOLF, L. *On projective spaces and resolutions in categories of completely regular spaces* Colloq. Math. 18 (1967) 185-196

[141] ______________ *H-closed and extremally disconnected Hausdorff spaces* Rozprawy Mat. 66 (1969)

[142] MISHRA, A.K. *Spaces in which chain-nets suffice* To appear.

[143] MITCHELL, B. Theory of Categories Academic Press, 1965

[144] MITYAGIN, B.S. and SHVARTS, A. *Functors in categories of Banach spaces* Russian Surveys 19 No.2 (1964) 65-127

[145] MIZUMOTO, M., TOYODA, J., and TANAKA, K. — *Some considerations on Fuzzy automata* J. Comp. Syst. Sci. 3 (1969) 409–422

[146] NEGREPONTIS, J. — *Duality in analysis from the point of view of category theory* Notices AMS 17 (1970) 280

[147] NYIKOS, P. — *Not every o-dimensional real compact space is N-compact* Talk in the Pittsburgh Conference on General Topology 1970

[148] PARK, Y.M. — *On the projective cover of the Stone-Cech compactification of a completely regular Hausdorff space* Canad. Math. Bull. 12 (1969) 327–331

[149] PERVIN, W.J. — *Quasiuniformization of topologic spaces* Math. Ann. 147 (1962) 316–317

[150] PONOMAREV, V.I. — *On the absolute of a topological space* Soviet Math. Dokl. 4 (1963) 299–302.

[151] POPA, M. — Quelques application de la theorie des categories dans distributiones. Rev. Roum Math. Pures et Appl. 13 (1968) 671-682

[152] __________ — Sur les support des distributiones. ibid 14 (1969) 1055-59

[153] POTHOVEN, K. — Projective and injective objects in the category of Banach spaces. Proc. AMS 22 (1969) 437-438

[154] PREUSS, G. — Eine charakterisierung epireflectiver unterkategorien einer Kategorie mit Inversionseigenschaft. Manuscripta Math 1 (1969) 307-316

[155] PUPPE, D. — Stabile Homotopietheorie. Math. Ann. 169 (1967) 243-274

[156] RAINWATER, J. — A note on projective resolutions. Proc. AMS 10 (1959) 734-735

[157] ROSEN, R. — The representation of biological systems from the stand-point of the theory of categories. Bull. Math. Biophysics 20 (1958) 317-341

[158] SCHLOMIUK, D. An elementary theory of the category of topological spaces
Trans. AMS 149 (1970) 259-278

[159] SCHNORR, C.P. Transformational classes of grammers
Inf. and control 14 (1969) 252-277

[160] SEMADENI, Z. Projectivity, injectivity and duality
Rozprawy Mat. 35 (1963)

[161] ____________ Categorical methods in convexity
281-307 in Proc. Colloq. convexity Copenhagen 1965

[162] ____________ Categorical approach to extension problems
193-198 in Proc. Symp. Ext. Theory Top. Struct., Berlin 1967 Academic Press.

[163] SKULA, L On a reflective subcategory of the category of topological spaces
Trans. AMS 142 (1969) 37-41

[164] STONE, M.H. Applications of the theory of Boolean rings to general topolog
ibid 41 (1937) 375-481

STRAUSS, D. Extremally disconnected spaces
Proc. AMS 18 (1967) 305-309

] TAYLOR, J. Weak families of maps
Canad Math. Bull 8 (1965) 771-781

67] VELICKO, N. Extensions of mappings of topological spaces
Soviet Math. Dokl 8 (1967) 1562-65

[168] WALTERS, R. An alternative approach to universal algebra
64-74 in Lecture Notes in Math. 106.
Springer-Verlag 1969

[169] WARD, F. On free and projective topological groups
Notices AMS 17 (1970) 135

[170] WILBER, W.J. Reflective subcategories and the generation of locally convex spaces
Notices AMS 18 (1971) 137

[171] WRIGHT, J. An extension theorem and a dual proof of a theorem of Gleason
J. London Math. Soc. 43 (1968) 699-702

[165] STRAUSS, D. Extremally disconnected spaces
Proc. AMS 18 (1967) 305-309

[166] TAYLOR, J. Weak families of maps
Canad Math. Bull 8 (1965) 771-781

[167] VELICKO, N. Extensions of mappings of topological spaces
Soviet Math. Dokl 8 (1967) 1562-65

[168] WALTERS, R. An alternative approach to universal algebra
64-74 in Lecture Notes in Math. 106.
Springer-Verlag 1969

[169] WARD, F. On free and projective topological groups
Notices AMS 17 (1970) 135

[170] WILBER, W.J. Reflective subcategories and the generation of locally convex spaces
Notices AMS 18 (1971) 137

[171] WRIGHT, J. An extension theorem and a dual proof of a theorem of Gleason
J. London Math. Soc. 43 (1968) 699-702

[172] WYLER, O. *A survey of top categories* Carnegie-Mellon University Res. Report 70-26

[173] ____________ *Top categories and categorical topology* Gen. Top. and Appl. 1 (1971) 17-28

[174] ____________ *An unpleasant theorem about limit spaces* To appear

[175] YANG, K.W. *Completions of normed linear spaces* Proc. AMS 19 (1968) 801-806

[176] YOUNG, G.S. *The introductions of local connectivity by change of topology* Amer. J. Math. 68 (1946) 479-497

[177] ZADEH, L. *Fuzzy sets* Inf. and control 8 (1965) 338-353

[178] ZILBER, J. Categories in homotopy theory Ph.D. Thesis, Harvard 1963.